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ON SOME MULTIVARIABLE GENERALIZED TRUESDELL POLYNOMIALS

R. C. SINGH CHANDEL & S. S. CHAUHAN

ABSTRACT : Motivated by the work of Chandel and Tiwari [1] and Singh [2], here in the present paper we introduce and study multivariable generalized Truesdell polynomials defined by (1.3). Its further generalization will also be discussed.

Key words : Generalized Truesdell polynomials, multivariable analogues, Generating relations, Rodrigues formula, Pure Recurrence relations, Differential Recurrence Relations.
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1. INTRODUCTION

Chandel and Tiwari [1] introduced multivariable analogue of Hermite polynomials defined by Rodrigues' formula :

$$(1.1)\qquad H_{n_1,\ldots,n_m}^{(b,h_1,\ldots,h_m,r_1,\ldots,r_m)}(x_1,\ldots,x_m)=(-1)^{n_1+\;\;+n_m}\left(1+h_1x_1^{r_1}+\ldots+h_mx_m^{r_m}\right)^b$$

$$\frac{d^{n_1}}{dx_1^{n_1}}\cdots\frac{d^{n_m}}{dx_m^{n_m}}\left(1+h_1x_1^{r_1}+\ldots+h_mx_m^{r_m}\right)^{-b}.$$

where $n_1,\ldots,n_m$ are positive integers, while $h_1,\ldots,h_m$, $r_1,\ldots,r_m$ are any numbers real or complex independent of $x_1,\ldots,x_m$.

Singh [2] introduced generalized Truesdell polynomials by the Rodrigues' formula :

$$(1.2)\qquad T_n^{\alpha}(x,r,p)=x^{-\alpha}e^{px^r}\delta^n\left(x^{\alpha}e^{-px^r}\right),\ \delta\equiv x\frac{\partial}{\partial x}$$

where n is positive integer and α, p, r are arbitrary numbers real or complex independent of x.

Motivated by the above work [1,2], here in the present paper, we introduce and study multivariable generalized Truesdell polynomials $T_{n_1,\dots,n_m}^{(b,\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots p_m)}(x_1,\dots,x_r)$ defined by Rodrigues' formula:

(1.3)
$$T_{n_1,\dots,n_m}^{(b;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots p_m)}(x_1,\dots,x_m)$$

$$= \left[1-\left(\alpha_1, \log x_1 - p_1x_1^{r_1}\right) - \dots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^b$$

$$\delta_1^{n_1}\dots\delta_m^{n_m}\left[1-\left(\alpha_1 \log x_1 - p_1x_1^{r_1}\right) - \dots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]^{-b}$$

where n_i are positive integers, b, α_i, r_i, p_i are arbitrary numbers real or complex independent of x and $\delta_i \equiv x_i \dfrac{\partial}{\partial x_i}; i = 1,\dots,m$.

It is clear that

(1.4)
$$\lim_{h\to\infty} T_{n_1,\dots,n_m}^{\left(b;\frac{\alpha_1}{b},\dots,\frac{\alpha_m}{b};r_1,\dots,r_m;\frac{p_1}{b},\dots,\frac{p_m}{b}\right)}(x_1,\dots,x_r)$$

$$= T_{n_1}^{\alpha_1}(x_1,r_1,p_1)\dots T_{n_m}^{\alpha_m}(x_m,r_m,p_m)$$

where $T_n^{\alpha}(x, r, p)$ are generalized Truesdell polynomials of Singh [2] defined by (1.2).

2. SOME FAMILIAR PROPORTIES OF $\delta \equiv x\dfrac{\partial}{\partial x}$.

Here we write some familiar properties of δ, which will be useful in our investigations:

(2.1) $\delta^n\{x^{\alpha}\} = \alpha^n x^{\alpha}$,

(2.2) $\delta^n\{u.v\} = \sum_{k=0}^{n}\binom{n}{k}\delta^{n-k}\{u\}\delta^k\{v\}$,

(2.3) $F(\delta)\{e^{g(x)}f(x)\} = e^{g(x)}F(\delta + xg')\{f(x)\}$

and

(2.4) $e^{t\delta}\{f(x)\} = f(xe^t)$.

3. GENERATING RELATION.

Starting from defination (1.3), we have

$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b,\alpha_1,\ldots,\alpha_m,r_1,\ldots,r_m,p_1,\ldots,p_m)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^b$$

$$e^{+(t_1\delta_1+\ldots+t_m\delta_m)}\left\{\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^{-b}\right\}$$

Now making an appeal to (2.4), we derive the generating relation:

(3.1)
$$\sum_{n_1,\ldots,n_m=0}^{\infty} T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m,r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$=\left[1-\left(\alpha_1\log x_1-p_1x_1^{r_1}\right)-\ldots-\left(\alpha_m\log x_m-p_mx_m^{r_m}\right)\right]^b$$

$$\left[1-\left\{\alpha_1(\log x_1+t_1)-p_1x_1^{r_1}e^{r_1t_1}\right\}-\ldots-\left\{\alpha_m(\log x_m+t_m)-p_mx_m^{r_m}e^{r_mt_m}\right\}\right]^{-b}$$

4. APPLICATIONS OF GENERATING FUNCTION.

Through generating function (3.1), we derive

(4.1)
$$T_{n_1,\ldots,n_m}^{(b+b';\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$=\sum_{k_1=0}^{n_1}\cdots\sum_{k_m=0}^{n_m}\binom{n_1}{k_1}\cdots\binom{n_m}{k_m}T_{k_1,\ldots,k_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$T_{n_1-k_1,\ldots,n_m-k_m}^{(b';\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

5. PURE RECURRENCE RALATIONS

Differentiating (3.1) partially with respect to t_1 and equating the coefficients of $t_1^{n_1} \dots t_m^{n_m}$ both the sides, we obtain pure recurrence relation:

$$(5.1) \qquad \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \dots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]$$

$$T_{n_1+1,n_2,\dots,n_m}^{(b;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= b\alpha_1 T_{n_1,n_2,\dots,n_m}^{(b+1;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-bp_1 r_1 x_1^{r_1} \sum_{k=0}^{n_1} r_1^{k} \binom{n_1}{k} T_{n_1-k,n_2,\dots,n_m}^{(b+1;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m);$$

which suggests m results in the following unified form :

$$(5.2) \qquad \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \dots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]$$

$$T_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}^{(b;\alpha_1,\dots,\alpha_m;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m)}(x_1,\dots,x_m)$$

$$= b\alpha_i T_{n_1,\dots,n_m}^{(b+1;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-bp_i r_i x_i^{r_i} \sum_{k=0}^{n_i} r_i^{k_i} \binom{n_i}{k_i} T_{n_1,\dots,n_{i-1},n_i,n_{i+1},\dots,n_m}^{(b+1;\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m); \qquad (i = 1,\dots, m)$$

6. DIFFERENTIAL RECURRENCE RELATIONS.

Differentiating (3.1) partially with respect to x_1 and equating the coefficients of $t_1^{n_1} \dots t_m^{n_m}$, we establish

$$(6.1) \qquad \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \dots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]$$

$$\frac{\partial}{\partial x_1} T_{n_1,\ldots n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= -\frac{b}{x_1}\left(\alpha_1 - p_1 r_1 x_1^{r_1}\right) T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m);$$

$$+\frac{b\alpha_1}{x_1} T_{n_1,\ldots,n_m}^{(b+1;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$-b p_1 r_1 x_1^{r_1-1} \sum_{k=0}^{n_1} \binom{n_1}{k} r_i^{k_i} T_{n_1-k,n_2,\ldots,n_m}^{(b+1;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

which can be further written as

$$\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right] \tag{6.2}$$

$$\delta_1 T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= -b\left(\alpha_1 - p_1 r_1 x_1^{r_1}\right) T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m,r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$+b\alpha_1 T_{n_1,\ldots,n_m}^{(b+1;\alpha_1,\ldots,\alpha_m,\ldots,r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$-b p_1 r_1 x_1^{r_1} \sum_{k=0}^{n_1} \binom{n_1}{k} r_1^{k} T_{n_1-k,n_2,\ldots,n_m}^{(b+1;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

This result further suggests m differential recurrence relations in the following unified form:

$$\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right] \tag{6.3}$$

$$\delta_i T_{n_1,\ldots,r_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= -b\left(\alpha_i - p_i r_i x_i^{r_i}\right) T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$+b\alpha_1 T_{n_1,\ldots,n_m}^{(b+1;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$-b p_i r_i x_i^{r_i} \sum_{k=0}^{n_i} \binom{n_i}{k} r_i^{k} T_{n_1,\ldots,n_{i-1},n_i-k,n_{i+1},\ldots,n_m}^{(b+1;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m) \quad i = 1,\ldots,m.$$

An appeal to (5.2) and (6.3) further shows that

$$\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right] \tag{6.4}$$

$$T_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$-\left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]$$

$$\delta_i\left\{T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\}$$

$$= -b\left(\alpha_i - p_i r_i x_i^{r_i}\right) T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m);\quad i = 1,\ldots,m.$$

which can be further written as

$$\left[b\left(p_i r_i x_i^{r_i} - \alpha_i\right) + \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right]\delta_i\right] \tag{6.5}$$

$$T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= \left[1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)\right].$$

$$T_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\quad i = 1,\ldots,m,$$

that is

$$\left(\frac{b\left(p_i r_i x_i^{r_i} - \alpha_i\right)}{1-\left(\alpha_i \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)} + \delta_i\right) \tag{6.6}$$

$$T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= T_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\quad i = 1,\ldots,m.$$

Now for brevity, we consider the operator

$$(6.7)\qquad \mathcal{D}_i \equiv \frac{b\left(p_i r_i x_i^{r_i} - \alpha_i\right)}{1-\left(\alpha_1 \log x_1 - p_1 x_1^{r_1}\right) - \ldots - \left(\alpha_m \log x_m - p_m x_m^{r_m}\right)} + \delta_i \quad (i = 1,\ldots,m).$$

Therefore, (6.6) reduces to

$$(6.8)\qquad \mathcal{D}_i\left\{T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\}$$

$$= T_{n_1,\ldots,n_{i-1},n_i+1,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m),\ (i = 1,\ldots,m),$$

which further shows that

$$(6.9)\qquad \mathcal{D}_i^k\left\{T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\}$$

$$= T_{n_1,\ldots,n_{i-1},n_i+k,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m) \quad (i=1,\ldots,m).$$

Specially for $k=n_j (j=1,\ldots,m)$, (6.9) gives

$$(6.10)\qquad \mathcal{D}_i^{n_j}\left\{T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\}$$

$$= T_{n_1,\ldots,n_{i-1},n_i+n_j,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,r_m) \quad (i=1,\ldots,m,\ j=1,\ldots,m).$$

Also

$$(6.11)\qquad \prod_{i=1}^{m}\mathcal{D}_i^{k_i}\left\{T_{n_i,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\}$$

Also from (6.9), we derive

$$(6.12)\qquad e^{t\mathcal{D}_i}\left\{T_{n_1,\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\}$$

$$= \sum_{k=0}^{\infty}\frac{t^k}{k!}T_{n_1,\ldots,n_i-1,n_i+k,n_{i+1},\ldots,n_m}^{(b;\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

7. FURTHER GENERALIZATION.

(7.1) $$G_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_r)$$

$$\left(G\left[\left(\alpha_1 \log x_1 - p_1 x_1^{n_1}\right) + \ldots + \left(\alpha_m \log x_m - p_1 x_m^{n_m}\right)\right]\right)^{-1}$$

$$\delta_1^{n_1}\ldots\delta_m^{n_m}\left\{G\left[\alpha_1 \log x_1 - p_1 x_1^{n_1}\right] + \ldots + \left(\alpha_m \log x_m - p_1 x_m^{n_m}\right)\right\}$$

where

(7.2) $$G(z) = \sum_{n=0}^{\infty} \gamma_n z^n, \quad \gamma_0 \neq 0$$

which for $\gamma_n = \frac{(b)_n}{n!}$, reduces to (1.3).

white for $\gamma_n = \frac{(-1)^n}{n!}$ (7.1) defines the multivariable polynomials as

(7.3) $$E_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_r)$$

$$= x_1^{-\alpha_1}\ldots x_m^{-\alpha_m} \exp\left(p_1 x_1^{n_1} + \ldots + p_m x_m^{r_m}\right)\delta_1^{n_1}\ldots\delta_m^{n_m}$$

$$\left\{x_1^{\alpha_1}\ldots x_m^{\alpha_m} \exp\left[-\left(p_1 n_1^{r_1} + \ldots + p_m n_m^{r_m}\right)\right]\right\}$$

It is quite clear that

(7.4) $$E_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

$$= T_{n_1}^{\alpha_1}(x_1, r_1, p_1)\ldots T_{n_m}^{\alpha_m}(x_m r_m, p_m),$$

where $T_n^{\alpha}(x,r,p)$ are eneralized truesdell polynomial of Singh [2]

8. GENERATING RELATION.

Starting from (7.1), we have

$$(8.1)\qquad \sum_{n_1,\ldots,n_m=0}^{\infty} G_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_r)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= \frac{1}{G\left(\alpha_1\log x_1 - p_1x_1^{r_1}\right)+\ldots+\left(\alpha_m\log x_m - p_mx_m^{r_m}\right)} e^{t_1\delta_1+\ldots+t_m\delta_m}$$

$$\left\{G\left[\left(\alpha_1\log x_1 - p_1x_1^{r_1}\right)+\ldots+\left(\alpha_m\log x_m - p_mx_m^{r_m}\right)\right]\right\}$$

$$= \frac{1}{G\left(\alpha_1\log x_1 - p_1x_1^{r_1}\right)+\ldots+\left(\alpha_m\log x_m - p_mx_m^{r_m}\right)}$$

$$G\left[\left(\alpha_1\log\left(x_1e^{t_1}\right) - p_1\left(x_1e^{t_1}\right)^{r_1}\right)+\ldots+\left(\alpha_m\log\left(x_me^{t_m}\right) - p_m\left(x_me^{t_m}\right)^{r_m}\right)\right] \text{ [by (2.4)].}$$

Hence finally, we derive the generating relation:

$$(8.1)\qquad \sum G_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= \frac{G\left[\log\left(x_1^{\alpha_1}\ldots x_m^{\alpha_m}\right)+\left(\alpha_1t_1+\ldots+\alpha_mt_m\right)-\left(p_1x_1^{r_1}e^{r_1t_1}+\ldots+p_mx_m^{r_m}e^{r_mt_m}\right)\right]}{G\left[\log\left(x_1^{\alpha_1}\ldots x_m^{\alpha_m}\right)-\left(p_1x_1^{r_1}+\ldots+p_mx_m^{r_m}\right)\right]}$$

9. SPECIAL CASE

$E_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$

In this section, we shall discuss the results for special case (7.3) which can not be derived for general case (7.1)

By an appeal to (7.3), we have

$$\sum_{n_1,\dots,n_m=0}^{\infty} E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!}$$

$$= x_1^{-\alpha_1}\dots x_m^{-\alpha_m}\exp\left(p_1x_1^{n_1}+\dots+p_mx_m^{r_m}\right)e^{t_1\delta_1+\dots+t_m\delta_m}$$

$$\left\{x_1^{\alpha_1}\dots x_m^{\alpha_m}\exp\left(p_1x_1^{n_1}+\dots+p_mx_m^{r_m}\right)\right\}$$

$$= \exp\left[p_1x_1^{r_1}+\dots+p_mx_m^{r_m}+\alpha_1t_1+\dots+\alpha_nt_n-\left(p_1x_1^{r_1t_1}e^{r_1t_1}+\dots+p_mx_m^{r_m}e^{r_mt_m}\right)\right]$$

Thus generating relation is given by

$$\sum_{n_1=0,n_m=0}^{\infty} E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)\frac{t_1^{n_1}}{n_1!}\cdots\frac{t_m^{n_m}}{n_m!} \tag{9.1}$$

$$= \exp\left[\alpha_1t_1+\dots+\alpha_mt_m+p_1x_1^{r_1}\left(1-e^{r_1t_1}\right)+\dots+p_mx_m^{r_m}\left(1-e^{r_mt_m}\right)\right],$$

which further suggest that

$$E_{n_1,\dots,n_m}^{(\alpha_1+\alpha'_1,\dots,\alpha_m+\alpha'_m;r_1,\dots,r_m;p_1+p'_1,\dots,p_m+p'_m)}(x_1,\dots,x_m) \tag{9.2}$$

$$= \sum_{k_1=0}^{n_1}\dots\sum_{k_m=0}^{n_m}\binom{n_1}{k_1}\dots\binom{n_m}{k_m}E_{n_1-k_1,\dots,n_m-k_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$E_{k_1,\dots,k_m}^{(\alpha'_1,\dots,\alpha'_m;r_1,\dots,r_m;p'_1,\dots,p'_m)}(x_1,\dots,x_m)$$

Also from (9.1) we can derive

$$E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m) \tag{9.3}$$

$$= \sum_{k_1=0}^{n_1}\dots\sum_{k_m=0}^{n_m}\binom{n_1}{k_1}\dots\binom{n_m}{k_m}(\alpha_1-\beta_1)^{k_1}\dots(\alpha_m-\beta_m)^{k_m}$$

$$E_{n_1-k_1,\dots,n_m-k_m}^{(\beta_1,\dots,\beta_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

Now differtiating (9.1) partially with respect to t_1 and equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$, we obtain pure recurrence relation

$$(9.4)\qquad E_{n_1+1,n_2,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= \alpha_1 E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-r_1 p_1 x_1^{r_1} \sum_{k=0}^{n_1} r_1^k \binom{n_1}{k} E_{n_1-k,n_2,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

which further suggests m results in the following unified form:

$$(9.5)\qquad E_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= \alpha_i E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-r_i p_i x_i^{r_i} \sum_{k=0}^{n_i} r_i^k \binom{n_i}{k} E_{n_1,\dots,n_{i-1},n_i-k,n_{i+1},\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m) \qquad i = 1,\dots,m.$$

Now diffretiating (9.1) partially with respect to x_1 and equating the coefficients of $t_1^{n_1}\dots t_m^{n_m}$ both the sides, we establish diffential recurrence relation

$$(9.6)\qquad \frac{\partial}{\partial x_1} E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= p_1 r_1 x_1^{r_1-1} E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$-p_1 r_1 x_1^{r_1-1} \sum_{k=0}^{n_i} r_1^k \binom{n_1}{k} E_{n_1-k,n_2,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

That is

$$(9.7)\qquad \left[p_1 r_1 x_1^{r_1} - \delta_1\right] E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots;p_m)}(x_1,\dots,x_m)$$

$$= p_i r_i x_i^{r_i} \sum_{k=0}^{n_i} r_i^k \binom{n_1}{k} E_{n_1-k,n_2,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m), \quad \delta_1 \equiv x_1 \frac{\partial}{\partial x_1},$$

which suggests m-differentional recurrence relations in the following unified form:

(9.8)
$$\left[p_i r_i x_i^{r_i} - \delta_i\right] E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= p_i r_i x_i^{r_i} \sum_{k=0}^{n_i} r_i^k \binom{n_1}{k} E_{n_1,\dots,n_{i-1},n_1-k,n_{i+1},\dots n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$\delta_i \equiv x_i \frac{\partial}{\partial x_i},\ i = 1,\dots,m.$$

By (9.5) and (9.8), we derive

(9.9)
$$E_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

$$= \left[\alpha_i - p_i r_i x_i^{r_i} - \delta_i\right] E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

For brevity, consider

(9.10)
$$\mathcal{D}_i = \alpha_i - p_i r_i x_i^{r_i} + \delta_i$$

Therefore

(9.11)
$$\mathcal{D}_i\left\{E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)\right\}$$

$$= E_{n_1,\dots,n_{i-1},n_i+1,n_{i+1},\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

from which we can further write

(9.12)
$$\mathcal{D}_i^k\left\{E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)\right\}$$

$$= E_{n_1,\dots,n_{i-1},n_i+k,n_{i+1},\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)$$

Hence

(9.13)
$$E^{t\mathcal{D}_i}\left\{E_{n_1,\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)\right\}$$

$$= \sum_{k=0}^{\infty} E_{n_1,\dots,n_{i-1},n_i+k,n_{i+1},\dots,n_m}^{(\alpha_1,\dots,\alpha_m;r_1,\dots,r_m;p_1,\dots,p_m)}(x_1,\dots,x_m)\frac{t^k}{k!}, \qquad i = 1,\dots,m.$$

Also

$$\mathcal{D}_i^{k_1}\ldots\mathcal{D}_m^{k_m}\left\{E_{n_1,\ldots,n_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)\right\} \tag{9.14}$$

$$= E_{n_1+k_1,\ldots,n_m+k_m}^{(\alpha_1,\ldots,\alpha_m;r_1,\ldots,r_m;p_1,\ldots,p_m)}(x_1,\ldots,x_m)$$

REFERENCES

1. Chandel, R.C.S. and Tiwari, A., multivariable analogue of Hermite polynomials, *Ganita Sandesh,* 5 (1991), 92-95.
2. Singh, R.P., On generalized Truesdell polynomials, *Riv. Mat. Unvi. Parma* (2) 8 (1967) 345-353.

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CONCEPT OF ŚŪNYA IN INDIAN ANTIQUITY

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ABSTRACT : The genesis of 'zero' as a number, that even a child so casually uses today, is a long and involved one. A substantial number of persons concerned with the history of its evolution, today accept that the number 'zero'—in its true potential, as we use it in our present day mathematics—has its root, conceptually as well as etymologically, in the word 'Śūunya' of Indian antiquity. However, the time-frame of its origin is still hotly debated. Furthermore, some recent works even try to suggest that a trace of the concept, if not in total operational perspective, might have a Greek origin that traveled to India during the Greek invasion of the northern part of the country in the pre-Mauryan period.

The present article discusses the concept of 'Śūunya' and the words akin to it, in its broader social and philosophical contexts, which eventually paved the path for the evolution of the corresponding mathematical concept. It appears very likely that the thread of rich philosophical and socio-academic ambiances of Indian antiquity was quite pregnant with the immensity of the concept of 'Śūunya'—a dichotomy as well as a simultaneity between nothing and everything, the 'zero' of void and that of an all-pervading 'fathomless' infinite.

This paper is a brief survey, for the sake of mathematical community, the present state of affairs through various claims and counter claims of the different schools of History of Mathematics, about the possible origin of present day mathematical concept of 'zero' along with its symbol, during the Indian antiquity.

Keywords : Śūunya, kha, zero, Hellenistic zero, chandaḥsūtra, lopaḥ.

1. INTRODUCTION

"*History is, by definition, the period for which we have written sources. In this sense, the history of Western civilization begins roughly about 3000 BC. In its 5000-year history, different nations have, over time, occupied centerstage by virtue of intense displays of complex activity*"—so observed Leo Depuydt,[13] a noted historian of Antiquity. Going by such a 'definition', one may easily be tempted to reach a conclusion as follows:"Our current numbering system is Indian in origin and took shape during the period of cultural and intellectual splendour that took place along the valley of the Ganges from the mid-III century to the mid-VI century AD. During this period, the Gupta dynasty reigned the region."[15]

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The aforesaid 'definition' cannot be of much help and needs to be taken with a pinch of salt, if one were to trace the genesis of mathematical knowledge in Indian Antiquity, particularly in the pre-Christian era, perhaps dating back to several millennia further, owing to the complete unavailability of written documents pertaining to this period. However, from the huge collection of available documents of later dates[1], beginning from the early Christian era, where one may find regular references to ancient works of amazingly lofty philosophical depth of thought, mingled with occasional flashes of extraordinary scientific sophistication, it goes without saying that the remote antiquity in India cradled a civilization of a very high order. As Sri Aurobindo observes "*Power of the ancient Indian spirit was a strong intellectuality, at once austere and rich, robust and minute, powerful and delicate, massive in principle and curious in details*".[3] It is with this background of the social context and ambiance of the Indian Antiquity in mind, that one must try to judge the possible potential mathematical developments, as testified passively by various apparently non mathematical resources, in case of the absence of direct mathematical testimony. In this article we shall try to throw some light in this direction, with an intention to illuminate some of the mystic and shadowy corners of this great culture.

In a recent scholarly book Kaplan[34] observes: "*Perhaps the world would be better and the past more attractive if some dead Indian had devised the hollow circle for zero rather than some dead Greeks (though I can't see why, especially since the concept matters more than its marker, and the concept, as we have seen, goes back to some dead Babylonians).*"

However, the Babylonian place-value system was sexagesimal, or based on the number 60. Some historians believe that its influence in India is found in the later astronomical works, (though this hypothesis is not widely accepted) but not in early India[42]. And the Babylonian symbol of zero, found inscribed in their clay tablets, denoted by two slanted wedge signs, as one may find in Labat's *Manuel D'Épigraphie Akkadienne* Paris (1948) or Kugler's *Babylonische Mondrechnung*, Tafel II (and its transcription, pp.34), (1900) was only a placeholder, sign of separation, which was never used alone as a number in its own right.[25] There is no zero in the scientific texts of the first Babylonian Dynasty, and the figure is hardly attested in any texts prior to third century BCE[25](pp. 153). Mathematicians seem to have used the zero in the intermediate position of a number only, but the astronomers used it not only in the middle but also in the final and initial positions of numerical expressions[2]. Furthermore, the double wedge symbol for zero had the meaning of "void", but it does not appear to have been imagined as "nothing". Ifrah [25] cites certain examples of mathematical tablets from Susa depicting cases where numerical calculations were being done. When the outcome of a calculation resulted in a clearcut 'zero', the scribe seemed to be at a loss, and instead of inscribing their slanted double wedge there, either left the place vacant, or had recourse to expressions like "*the grain is finished.*" This allows us to conclude, following Ifrah[25], that "*the concept of 'void' and 'nothing' both existed* (in Babylonian mathematics). *But they were not yet seen as synonyms.*"

However, we are fortunate to have ample inscriptions on clay tablets bearing, among other things, monetary documents, testifying to the mathematical knowledge and skill of that period, from as early as pre-Sargonic era (the first half of the third millennium BCE), that these ancient Sumerians have bequeathed to us. It is a pity that no such document of early Indian Antiquity is extant. Before we begin to interpret the thoughts and history of this bygone era, we humbly recall Baruch Spinoza (1632-1677) : '*Not to ridicule, not to mourn, never to detest, but try to understand*'[3].

2. HISTORIOGRAPHIC PROBLEMS

"*Some of the chief historiographic difficulties surrounding Indian mathematics traditions include the fact that, the European scholars who encountered Indian mathematical texts in the eighteenth and nineteenth centuries were often completely at sea concerning the ages of the text, their interrelationship and even their identities. The sheer number of such works and the uncertainty surrounding the most basic chronology of the Sanskrit literature gave rise to great confusion, much of which survives to this day in discussions on Indian mathematics.*"[48]

Though some pioneering contributions of the Indian mathematicians of ancient period are generally acknowledged, there are reasons to believe that many famous historians of mathematics, while trying to find their way through this bewildering maze of extant Sanskrit texts towards establishing the exact nature and content of the Indian contribution, with a special focus on their period of inception, "*found what they expected to find, or perhaps even what they hoped to find, rather than to realize what was so clear in front of them.*"[43]

As a matter of fact, a good volume of mathematical works available for scholarly scrutiny is embedded in the contexts of medieval Indian astronomy and astrology, not naturally familiar to the European scholars. Furthermore, the presentation of these texts is usually found to be in highly compressed Sanskrit verses, frequently cryptic to the point of being obscure, and that too, not always in the present form of Sanskrit, but in what is called the Vedic Sanskrit, alien in appearance to a non specialist. "*Further, many texts, even on technical subject like jyotiṣa, were ascribed to the revelations of gods or legendary sages. This attribution expunged the historical contexts of the works to stress the divine importance of their content. Similarly, even historical human authors frequently omitted biographical information and other contextual details as irrelevant and unnecessary. This sometimes makes it difficult to distinguish between human and allegedly divine authors.*"[48] All these difficulties took their toll on the early surveys of Indian sources, "*which often tended to portray the Indian mathematical tradition as a record of discoveries or contributions, classified according to modern mathematical categories and important in proportion to their originality or priority.*" The context for understanding Indian mathematics in its own right, as an integral part of the social and cultural ambiance of Indian Antiquity, was generally neglected. There was yet another menace. "*The historiography of science in India has long been co-opted for political purposes. Most* *notoriously, some nineteenth century colonial officials disparaged local intellectual traditions, which they termed "native learning", in order to justify the westernized education for future colonial servants.*"[48] Adding to the injury is the fact that much of the desired data is simply unavailable from India's historical records, as far as the present state of information in this regard is concerned. Plofker[48] also points out, "*...it is only around the middle of the first millennium CE that one finds the first surviving complete Sanskrit text in the medieval tradition of mathematical astronomy*"— quite a frustrating state of affairs, more so, as it is known beyond doubt, that several surprisingly advanced civilizations with rich philosophical, linguistic and scientific culture flourished in India, predating the Christian era by millennia. This unfortunate information vacuum has given rise to an almost chaotic condition with myths and controversies, claims and counter claims, disputed 'proof's and fallacious presumptions. At one extreme we find the conscious shaping up of the history according to some pre-conceived 'Eurocentric' bias amounting to the doctrine aptly summarized by R. Rashed as "*Classical science is European and its origins are directly traceable to Greek philosophy and science*" and at the other we have the "*bandwagon of nationalist historiographers, who are obsessed with establishing the Indian priority in every aspect in the history of ideas*" [55], often putting forward extravagant claims to the autonomy or antiquity of their scientific tradition.[4] They present some evidences as irrefutable proof which the other group promptly rejects as forgeries. However, many of the generally accepted conclusions are nonetheless not definitely proved, and many revisionists or minority views have achieved widespread popularity. As a result, a shift in the focus of opinions regarding certain claims and counter-claims in this very dynamic and vibrant area of study has become quite commonplace. As for example, one may cite the issue of originality of the parameters of the planetary revolution of Āryabhaṭa, which is nowadays generally accepted as being independent of the Greek sources.[5] However, it has been strongly debated in a recent book by Plofker[48] [pp. 115-120][6]. However, the crux of the historiographic enigma regarding the mathematical sciences in Indian Antiquity is ultimately poised at the questions of when, how and by whom these texts were composed, and "*these questions still have no universally accepted answers in most of the cases.*[7]"

3. ENIGMA OF MIGRATION

While talking about 'Indian Antiquity', which refers to the known period of history up to around 6th century A.D., it is important to realize that, the group of inhabitants it refers to, particularly in the remote antiquity is still somewhat shrouded in mystery. Who were these people, supposedly so preoccupied with the pursuit of timeless spiritual knowledge, rather than the recording of mundane events of daily social life? This could have been called their 'history'— a term, which so rarely, if at all, can be ascribed to any available literature belonging to the vast corpus of Sanskrit texts. Where did they come from, or did they originate here, in India? However, it must be kept in mind that, while referring to India in antiquity, as per common norms, one takes into account the entire geographical locus of the present day nations of

Pakistan, Nepal, Bangladesh, Sri Lanka, along with present-day India. Analysing the available references from their spoken language, an archaic form of Sanskrit, in which the earliest known texts, called The Vedas, were composed, a school of linguists categorise them as Proto-Indo-European, which refers to various Indo-European themes and motifs found common in this culture to that of some ancient European ones. However, it is observed that the origin and diffusion of the common ancestral Indo-European cultures are still quite problematic. [48] It is generally proposed that a group of peoples, originally belonging to that culture, once came down to the Indian soil, through the high-altitude mountain passes of the Himalayas and eventually got settled there. This theory of the origin of Vedic Aryans, is usually referred to as the *Arian Invasion Theory*, or as the *Arian Immigration theory*, as some might prefer to call it. However, there are numerous difficulties with most of the features of this hypothesis. [48]

Contesting this hypothesis, one has the so-called *Indigenous Aryan Theory*, where one has no need to link the Vedas and their language to a hypothetical Indo-European expansion over the Afgan highlands. This theory claims that the prehistoric agrarian-urban culture of Indus Valley civilization, centered at Harappa, identified during an excavation in 1921-22 and Mohenjo-Daro, found around the same time, which flourished during the mid-third millennium BCE near the Indus river around Sind and Punjab, was the Vedic Culture in reality. Comparatively recent excavations, in nearby locations, have revealed sites like Mehrgarh, (excavation started in 1974, under a French archaeologist, J.F. Jarrige, near Bolan pass which is about 150 km from the Pakistani city of Queta) which is believed to have cradled an urban civilization that dates back to 7000 BCE. It appears that one can claim with confidence that the extraordinary advancements found in various facets of the Harappan civilization like agriculture, architecture, manufacture and trade were not a sudden development, but a gradual culmination of these various other folk cultures and urban civilizations, predating it by quite a few millennia.

Yet another hypothesis, called Out of India, not conforming much to the linguistic angle, suggests that instead of a few Indo-Europeans coming down to the subcontinent through the mountain passes, perhaps most Indo-Europeans had gone out of it! However, this theory also has several other weak points, as the evidences of Vedic Sanskrit culture, mostly textual, do not tally with that of the Indus Valley and related ones. These are mostly archaeological in nature, and hence difficult to compare, given that, in spite of numerous claims, a large collection of the Harappan epigraphs is still undeciphered[20]. Indeed, even this much is not known with certainty, whether they form a part of some ancient script or are merely nonlinguistic signs. However, early Vedic hymns do not refer to cities or wheat, well known in the Indus culture. On the other hand, archaeological excavations did not reveal remains of some very common characteristic Indo-European items like horses or chariots. Though analysing the evens and odds of these hypotheses, the *Arian Immigration theory* is presently considered to be the standard accepted narrative of the origin of the Vedic Civilization, one must categorically point out that, there is little definite evidence concerning various aspects of this theory. [48] It is also worth mentioning that, it is not known whether the ancient civilizations of Mohenjo-Daro and Harappa still existed when India was 'invaded' by the Aryans, or whether their writing had developed during this period[8].

4. EARLY VEDIC PERIOD

The sacred Vedas, whose name has the same root as that of the word *vidyā* which literally means 'knowledge',[9] are often considered the fountainhead of learning, preserving the knowledge and doctrines of remote antiquity.[48] The corpus of 'Vedic Texts' includes the four *saṁhitās*, the oldest layer (c. 1000 BCE) of Vedic literature. They are collections of hymns and rituals—namely the *Ṛgveda*, *Yajurveda*, *Sāmaveda* and *Atharvaveda*, of which the *Ṛgveda* is the oldest. The next phase of Vedic literature, consisting of the commentaries on *saṁhitās* are found in the *Brāhmanas* (c.800 BCE) and *Āraṇyakas* (c. 700 BCE), as well as explanatory and philosophical works in *Upaniṣads* (c. 600-500 BCE)[10]. One can find some references to the study of mathematics as a branch of *aparāvidyā*, the worldly knowledge, in the *Muṇḍaka Upaniṣad* and *Chāndogya Upaniṣad*. In the first millennium BCE, the division of learning included not only the Vedic texts themselves but also the six auxiliary limbs known as *Vedāṅgas* or *ṣaḍṅgas*. These supporting disciplines of the Vedas, "*were composed due to a conscious and meticulous effort towards preservation of the meaning and external forms of the mantras of saṁhitas*".[16] These are,[11]

śikṣā i.e., phonetics, dealing with the description of vowels and consonants towards ensuring the most correct reading of the Vedas and unfolding their inner truth;

kalpa i.e., ritual practice, which specified the details of the various rites;

vyākaraṇa i.e., grammar, important for a correct understanding of the exact meaning of words through perfect utterance;

nirukta i.e., etymology, as complementary to the grammar, enabling one to discover the primary deity of a stanza of hymns which may bear the characteristic marks of more than one deity, thus helping to perform the sacrifice with perfection;

chandas i.e., prosody or poetic metrics, which ensured the proper preservation and comprehension of the archaic verses of the hymns; and

jyotiṣa i.e., astronomy and calendrics, which determined the proper times for performance of the rites.

In Pāṇinīya śikṣā we have the following verse:

Chandaḥ pādau tu vedasya hastau kalpo'tha paṭhyate
jyotiṣām ayanaṃ cakṣur niruktaṃ śrotam ucyate

śikṣā ghrāṅaṃ tu vedasya mukhaṃ vyākaraṇaṃ smṛtam
tasmāt sāṅgam adhīty aiva brahmaloke mahīyate.

Chandas are the feet of the Vedas, Kalpa the hands,
Astronomy the eyes and Nirukta the ears;
Śikṣā constitutes the nose of the Vedas, Grammar is the mouth.

All these *Vedāṅgas* may be divided into two groups, explanatory and customary. *Kalpa* and *jyotiṣa* fall into the second group while the others belong to the first. The Vedic texts are generally known as *śruti*, 'heard' via divine revelation, hence *apouruṣeya*; the limbs of the Vedas, on the other hand, are called *smṛti*, remembered from human tradition. In the post-Vedic era of what is known as Classical period, one finds an enhanced form of *jyotiṣa* that incorporated not only astrology but also computational methods in general, known as *gaṇita*, the science of calculation, i.e., mathematics. The exact sciences and most other branches of *smṛti* learning were called *śastras*, i.e., 'teachings'. In the *Vedāṅga jyotiṣa* one finds the mention of the supreme importance of mathematics depicted in the following verse:

yathā śikhā mayūrāṇāṃ nāgānāṃ maṇayo yathā
tadvadvedāṅgaśāstrāṇāṃ gaṇitaṃ mūrdhani sthitam

Like the crest of a peacock, like the gem on the head of a snake,
so is mathematics at the peak of all knowledge.

However, though the importance of mathematics is so ceremoniously pointed out at such an early stage of Indian Antiquity, unfortunately no exclusively mathematical work of the remote part of this period is extant, except perhaps, the *Śulbasūtras*[12]. Composed in Vedic Sanskrit, the works of this period are primarily religious in content, 'as it was believed that True knowledge of whatever sort was necessarily a part of the fundamental truth of the Vedas', but they tacitly embody a significant amount of mathematical knowledge, chiefly astronomical in classification, which was perhaps necessary to determine the exact timing for the rituals and rites to be performed. Again, the accurate construction of the altars (*vedi*) and fire-places *(agni-kuṇḍa)* for sacrificial rites was necessary and perhaps that paved the way for the development of geometry[13] that comes to us codified in the manuals, called *Śulbasūtras*, associated with the *Yajurveda*. Four of them, *Baudhāyana*, *Mānava*, *Āpastamba*, *Kātyāyana*,[14] form separate treatises, the rest are chapters or parts of chapters of the corresponding *Śrautasūtras*[2]. These are the most extant mathematical documents of Indian Antiquity known to us, as of now, the earliest one (*Baudhāyana*) among them being placed around c. 800 BCE, justified through its pre-*Paṇinian* linguistic character, and the latest one (*Kātyāyana*) being somewhere around 200 BCE[39][15]. However, there are strong evidences favouring the fact that the knowledge of *śulba* mathematics dates back to the *Saṁhitās* and *Brāhmaṇas*, and the *śulba* authors themselves '*emphasise that they were merely stating facts*' known to the earlier seers.[16] Regarding the tacit presence of mathematical concepts in religious texts, for example

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decimal nomenclature of numbers of astonishingly high order, not in figures of course, but with the names of the numbers, numerous examples can be found in many treatises dating back to the *Saṃhitās*—the earliest phase of Vedic literature,[18],[16], clearly predating 1000 BCE, even by moderate estimates.[42]

In the early Vedic society, veneration for Sanskrit exalted it as sacred speech, *Devabhāṣā*, "whose divinely revealed texts were meant to be recited, heard and memorized collectively rather than transmitted in writing".[48] Naturally it gave rise to the necessity that such texts be composed in formats that could be easily memorized. It was done in two modes, either through condensed prose aphorisms, called *sūtras*,[16] or in verse form, particularly in the classical period, beginning in the late first millennium BCE. With the passage of time, the second form of presentation brought in numerous synonyms of a single term, for example, a number name, coined from different socio-cultural perspectives[17] and invoking 'poetic license' these were used freely, presumably due to the metrical need of the form of composition. Naturally, the practice of according preference to the ease of memorization sometimes came in the way of ease of understanding. As a result, most of the treatises are found to be supplemented, at a later period of time, by one or more prose commentaries or notes, called *(bhāṣya)* or *ṭikā* respectively. Often more than one in number, these works were composed by different authors writing from time to time on the same original text, who took more care either over the need of preservation and proper interpretation of the past knowledge or improving upon the previously available version by including in it their own reflections and interpretations, thanks to which we have extremely useful information about the nature and content of the original texts, most of which are not extant.

5. ŚŪNYA IN VARIOUS FORMS OF PERCEPTION

In ancient India, the usage of the term śūnya had a much broader significance as is revealed from various extant texts of the Indian antiquity as opposed to the much narrower literally translated present day meaning attributed to it through the term '*zero*'. A study of this subject leaves one with a sense of wonder at the awesome depth and breadth of ancient Indian thought.

5.1 Śūnya in philosophical context

In various perspectives, the Sanskrit term *śūnya* does not only convey the sense of void or mere nothingness, but also reveals that of perfect fullness (i.e., *pūrṇa*)[18], a divine infinititude (i.e., *ananta*) through its association with the metaphysical idea of 'ether', the vast expanse of limitless space (found in a wide variety of terms like *kha*, *gagana*, *ākāśa*, *ambara*, *vyoma*, *antarikṣa*), and sometimes a receptive vacuum,'*the unformed and the essence of all that is uncreated*', like what happens after the destruction of the universe (i.e., *pralaya*) by *Śiva*—the eternal destroyer, a vacuum pregnant with the immensity of creation of a new infinite universe (i.e., *sṛṣṭi*), created out of the *śūnya* by *Śivā*—the eternal creator. "*The vacuum is* *truly a living void, pulsating with endless rhythms of destruction and creation*[19]", the duality between the minutest of the minute and the greatest of the great, the *Brahman*, perceived as *aṇor aṇīyān mahato mahīyān*[20].

Is the presence of nothing i.e., *śūnya* (*reflecting non-existence*), different from the absence (*akin to voidness*) of something or anything (*reflecting non-availability*)? [28] The Sanskrit word *śūnya*[21] is probably derived from *śūnā*, [as *śūnā* + *yat*], which is the past participle of √*śvi*, which means 'to swell' or,'to grow', and therefrom by semantic extension, 'hollow'. In *Ṛgveda*, one may find another meaning : 'the sense of lack or deficiency'[22]. "*It is possible that the two different words were fused to give śūnya a single sense of absence or emptiness with the potential for growth*"[28], a womb-like hollow, ready to swell. Its conceptual derivative, *śūnyatā*, the state of void, is described in the *Nāsadīyasūkta* of *Ṛgveda*[23]. According to the teaching of the *Tantra*, both *ātman* and *Śiva* are to be conceived as voids, '*undifferentiated formless entity*'[24]; in the first one (*ātman*) the divine qualities of *Śiva* have not flourished yet and the objective of 'this void' (*ātman*) is to attain the 'other void' known as *Śiva*.[8] In this context, one may also refer to the *Mādhyamika* school of the *Mahāyāna* Buddhist's doctrine of devoidness, (*śūnyavāda* i.e., nihilism, of the philosopher Nāgārjuna) who argued[25] in favour of the spiritual practice of emptying the mind of all impressions. According to this doctrine, the highest form of knowledge, *prajñāpāramitā*, is the perception of everything phenomenal as *śūnya*, the pure void. Complete *śūnyatā* is the same as *nirvāna*, which '*encompasses the end and the beginning*'. But how to attain this state of *śūnya*? According to *Nāgasena*, a philosopher of the same school, it is to be done by stepwise negating (subtracting or taking away) all the parts of a thing. It soon paved the path for the corresponding mathematical development of thought, particularly among the Buddhist school—"*just as emptiness of space is a necessary condition for the appearance of any object, the number zero, being no number at all, is the condition for existence of all numbers*".[28]

5.2 Śūnya in mathematical context

To discuss various facets of the mathematical *śūnya* in Indian Antiquity, the following perspectives are of paramount importance:

(i) to trace the concept of *śūnya*, within the frame of a place-value system;

(ii) the symbol(s) used for *śūnya* and

(iii) the fundamental mathematical operations with the digit *śūnya*, considered as a digit in its own right.

The first pursuit, an involved one, will be done at length separately in the next section.

Symbol(s) used for śūnya:

As far as the symbol for *śūnya* in early phase of Indian antiquity is concerned, if there was any, and we shall argue in the later sections that there is every reason to believe, not parochially, that there was, in all probability, some such symbol in vogue,[26] unfortunately no direct evidence such as inscriptions or written documents favouring this argument is extant. So here again, one has to have recourse to the indirect analysis gingerly, via the socio-cultural root, in search of possible lingual mention of any such symbol. The earliest symbol for *śūnya* in India perhaps was a bold dot (*bindu*[27]), referred to as *śūnya-bindu* in the extant literatures composed around the early centuries of the Christian era. The association between the concept of zero and its symbol is found in a literary work (*Vāsavadattā* by the celebrated Sanskrit fictionist *Subandhu*, (ca. 400 CE[28], however B.B. Datta[10], Kaplan[34] place it in 620 CE) :

> *"The stars shone forth, like zero dots (śūnya-bindavaḥ)—scattered in the sky as if on the blue rug, the Creator reckoned the total with a bit of the moon for the chalk"*[28].

This idea was dormant in the Vedic literatures as well. The word *kṣudra* (literally meaning, very small) is found in the *Atharvaveda*[29](XIX. 22.6 [51]) in comparison with *ṛca*, the positive and *ānṛca*, the negative numbers.[10][30] It seems quite natural that the word numerals equivalent to *śūnya* like *randhra* (meaning a hole) found in *Amarakośa* or its synonym *chidra*[31] (meaning a puncture mark), gradually got enfigured into the round symbol of *śūnya*, a solid circular dot to begin with, and then at a later stage only the peripheral circle, perhaps to save time required for, or to avoid the botheration of, darkening its interior.

The *Bakṣālī Manuscript*[32]—the most ancient mathematical document of Indian Antiquity known so far, was found in 1881 at a place called *Bakśālī*, about eighty km. north-east of Peshawar (now in Pakistan), by a farmer who dug it up while cultivating a land. The untitled manuscript, written on birch-bark, is composed in *Gāthā* language (a modified form of *prākṛta*) and the scripts used are recognised as an earlier type of *Śāradā*, once used in the Kashmir region of Indian subcontinent. In the seventy fragmentary leaves of the document, which is now extant, one finds numerous intricate calculations involving decimal place-value notation of numerals, including a solid dot[33] for the digit *śūnya*.[11] The time and purpose of composition of this text is widely debated. The work has never been quoted or mentioned in any other works and might even be just an exercise compiled by a scribe.[34] Estimates of the date of this work range from as early as the 2nd Century of BCE up to the 12th century CE; however, as per some recent writings, it is believed to be not much later than 7th century CE.[35]

As far as the first written undisputed[36] evidence (stone inscription) of the use of 'zero' as a decimal digit is concerned, depicted as a neat round circle and found in Indian soil, as of now, one may find it in the two stone inscriptions of Gwalior[37] (875/876 CE), discovered[38] in the nineteenth century.

S.C. Kak[31],[32],[33] while working on the origin of the sign of zero in India, explored possible links between *Brāhmi* numerals and some Indus symbols. He came to the conclusion that "*the Indus 5 and 10 became the corresponding Brāhmi signs*", from which he proposes to push back "*the ancestry of the zero sign to the third millennium BCE.*" However, since no decipherment of Indus scripts is yet accepted in general, this can be regarded as a minority view.

Mathematical operations with śūnya:

Apart from the *Bakṣālī* Manuscript, of which the time frame is hotly debated, many examples of two fundamental operations by the zero (referred to as *kha*, *śūnya*, *ambara*) viz. that of addition and subtraction can be found in *Pañchasiddhāntikā* of *Varāhamihira* (505 CE). However, these are described in language and not by symbols[10]. Clear mathematical computations involving zero along with the formulations of rules necessary for such calculations are found in *Brahmagupta's*[39] (b. 598 CE) work *Brāhmasphuṭasiddhānta* (completed in 628 CE). He treated 'zero' (referred to as *kha*) as a separate number, neither positive (*dhana*) nor negative (*ṛṇa*). In chapter seven, on *Gaṇita*, while dealing with the formulations of norms of operation among various types of numbers, he states in clear terms

'dhanayor dhanam ṛṇam ṛṇayor dhanarṇayor antaraṃ samaikye khaṃ',

which means, '[the sum of two] positive [number]s [is] positive, [that of two] negative [number]s [is] negative, [the sum of one] positive [number and one] negative [is their] difference, [the sum of] two equals [opposite in sign, is] zero'[60]

—a mathematical principle to the effect that $x - x = 0$, treating zero as a number in its own right. From his time onwards, one may find in the Sanskrit texts on mathematics (mainly devoted to Astronomy) a separate section entitled *śūnya-gaṇita*, i.e., computation with zero, which clearly indicates that the special status of zero in mathematical calculation was strongly established in the Indian academia by that time. Many more examples intermingled with the concept of place-value arithmetic are given in the next section.

6. DECIMAL PLACE VALUE AND ZERO

"*India gave (the) world a priceless gift—the decimal notation. This profound anonymous innovation is unsurpassed for sheer brilliance of abstract thought and utility as a practical invention. The decimal notation derives its power from two key strokes of genius : the concept of 'place-value', and the notion of zero as a digit.*"[16]

Undoubtedly, the concept of zero is most instrumental in making a place-value system work. It may apparently seem quite nonsensical an idea 'to have a symbol that represents nothing', but this idea, in harmony with the other numerical symbols[40] needed for the system, is precisely what makes a positional numbering i.e, a place-value based enumeration system work. Since there exist infinitely many numbers, to be able to write any one of them, however large, by merely using ten symbols (hence called the 'decimal' system, nine symbols for one to nine and the tenth one, for zero) was a truly a prodigious intellectual achievement of the Indian scholars of remote past.

It is worth quoting the famous French mathematician Pierre Simon De Laplace (1814) in this regard :

> *The ingenious method of expressing every possible number using a set of ten symbols (each symbol having a place-value and an absolute value) emerged in India. The idea seems so simple nowadays that its significance and profound importance is no longer appreciated. Its simplicity lies in the way it facilitated calculations and placed arithmetic foremost among useful inventions. The importance of this invention is more readily appreciated when one considers that it was beyond the two greatest men of antiquity, Archimedes and Apollonius.* [25]

The principle of place-value system assigns to each digit of a number a certain value (called the 'place-value') by virtue of its position (i.e., place) in that number. The decimal place value system, for example, assigns the place value one to the 'place' in the extreme right, often called the units place; the 'place' immediately left to it gets the value ten, followed by the 'places' of hundred, thousand, ten thousand and so on in a leftwardly direction. Clearly, each 'place' has a place-value which is a certain power of ten, hence the name *decimal.* For example, if we look at the number 2009 in this system, the digit (called *aṅka* in Sanskrit, which literally means 'mark') 2 having the intrinsic value two, stands for the value 'two thousand' by virtue of its being in the thousands place, while the digit 9, being in the units place, represents the value nine, which happens to be the intrinsic value of that digit as well. Observe that the two zeros standing in the middle, respectively occupying the tens and the hundreds place, do not apparently show up in the *number name* of 2009 (which is *two thousand nine*) but they do contribute to it, from behind the screen, in the sense that, it is their presence in the right 'place' that has allowed the digit 2 to occupy its correct 'place' for the decimal shaping of this number. This is an example of the powerful role of the digit zero as a placeholder. In the language of G.B. Halsted[21]: The importance of the creation of the zero mark can never be exaggerated.

> *This giving to airy nothing, not merely a local habitation and a name, a picture, symbol, but helpful power, is the characteristic of the Hindu race whence it sprang....No single mathematical creation has been more potent for the general on-go of intelligence and power.*

In a recent well-researched exposition A.K. Dutta[16] aptly poitned out that, "*In the decimal representation through words, terms of powers of ten plays the role of place-value principle*". Such concept can be found in the early Vedic text called *Vājasaneyī Saṃhitā* (XVII.2) of *Śukla Yajurveda* in a prayer of seer *Medhātithi*, where one can find the names of specific terms for each power of ten up to a billion, where each term is defined to be ten times the preceding one (for example, *daśa daśa* a *śata*, i.e., ten times ten is a hundred, *daśa śata a sahasra*, i.e., ten times hundred is a thousand, etc., going up to 10^{12}, called *parārdha*)[41].

One may further refer to Gupta[18] for similar examples in other *Saṁhitā* and *Brāhmaṇ a* texts. Instances of number names in conformity with decimal place-value representation can also be found in *Ṛgveda*—the oldest text among the Vedic corpus. Take for example, *sapta śatāni viṁśatiḥ* i.e., seven hundred twenty [I.164.11], and *sahasrāṇi śatā daśa* i.e., *one thousand one hundred ten* [II.1.8].[16]42 Ample evidences can be cited to claim that right in the era of the *Saṁhitās*, any number at least up to a billion, if not more, could be represented by word numerals verbally, using the nine word numerals [viz. eka (one), *dvi* (two), *tri* (three), *catur* (four), *pañca* (five), *ṣaṭ* (six), *sapta* (seven), *aṣṭa* (eight), nava (nine)] and the names indicating their position within a number in appropriate powers of ten. Dutta[16] observes "*to arrive at a written decimal notation from the above terminology, one has to simply suppress the place-names from a given numerical expression provided one has an additional tenth numeral as a placeholder to indicate the possible absence of the nine numerals in certain places. Thus one can conclude that the structure of the Sanskrit numeral system contains the key to the decimal place value system.*"

There are evidences to suggest that Buddhist scholars of ancient times were also well versed in the use of decimal place-value system and that knowledge of *Gaṇita* was considered important. The period from 500 BCE to 900 CE coincides with the rise and dominance of Buddhism. In the *Lalitavistāra*, biography of *Gautama Buddha* (b. 562 BCE), which may have been written around the first century CE, one finds *Buddha* enumerating to a mathematician *Arjuna*, upon being challenged by him, successive number names in powers of 10, beginning from a *koti* i.e., 10^7 up to 10^{53}, *tallakṣaṇa*.[43]

In *Vālmīki Rāmāyana*, while giving the exact strength of *Rāma*'s army, one may find (6.28.33-41) the use of *lakṣa* scale, in which the next term is 10^5 times of the previous one, giving terminologies of numbers up to 10^{55} (*mahaugha*) beginning from 10^7 (*koṭi*). A.K. Dutta has cited plenty of references[16] of place-value system being used in the various early Sanskrit texts, enumerating numbers up to as large as 10^{421}· Among them, one example of reference to decimal system is found to be in the great Hindu epic *Mahābhārata*. In the *Vanaparva* section, one finds a scholar, Bandi by name, debating with *Aṣṭāvakra* in the court of king Janaka saying : *navaiva yogo gaṇaneti śaśvat*, which means *nine is the perpetual number of symbols one combines in calculation*. B.B. Dutta (who later became known as *Swāmi Vidyāranya*), a pioneer among the historians of mathematics in early India, observed that, if the names involved in the aforesaid instance truly reflect the antiquity of the legend, then it will not be far-fetched to infer that the decimal place-value system was in use during the *Brāhmana* period of Vedic culture.[51] However, this does not form a strong evidence, as the exact time of original composition of this epic is uncertain and it is believed to have gone through several later alterations and additions.

The sufficiency of nine symbols, as mentioned above, if taken verbatim, may raise some eyebrows. Indeed, one has to presume that there was another symbol, denoting the absence of those nine, as otherwise, without such a notation, nine symbols would not have been adequate.

However, time and again, a school of scholars, sometimes referred to as the 'Eurocentric' school, (as according to their doctrine, the whole of the ancient mathematics is believed to have their European roots, be it in Greece or in Babylonia) have pointed out similar instances and examples[44], particularly pointing out the early Latin works/translations of the Arabic assimilations of Hindu works on mathematics. Some recent arguments in this direction may be found in Kaplan[34] [pp. 46].

A clear formulation of a rule pertaining to the decimal place-value based enumeration is found in the works of the famous *Āryabhaṭa* I (b. 476 CE)[45] who writes

ekaṃ daśa ca śatañca sahasramayutaniyute tathā prayutam
koṭyarbudañca bṛndaṃ sthānāt sthānaṃ daśaguṇaṃ syāt[46],

the last part of which can be translated as, (in a number) from one 'place' to the next, the place value is increased ten times.

Jinabhadra Gani (529-589) in *Bṛhatkṣetra-samāsa*, depicting a clear knowledge of zero in a place value system, gave the number 224400000000 as

dvi viṃśati ca catuścatvāriṃśati ca aṣṭa śūnyāni
'twenty-two and forty four and eight zeros'[25], [11]

Another example found in B.B. Datta, from the middle or end of the sixth century is of *Siddhasenagaṇi*. While commenting on *Tattvārthādhigama-sūtra* of *Umāsvāti*, a Jaina[47] mathematician of the *Śvetāmbara* sect, Siddhasena employed decimal place value numerals in his calculations, as has been elaborately shown in B.B. Datta[11]. Again, in the commentary by *Vyāsa* on *Patañjali*'s *Yogasūtra* (3.13) (7th century CE) one finds the following principle of enumeration:

yathaikā rekhā śatasthāne śataṃ daśasthāne daśai'kā caikasthāne....[42]
"just as the *rekhā* (a vertical line denoting one) connotes a hundred in the 'hundreds' place, ten in the 'tens' place and one in the 'units' place......."

In a recent encyclopedic work on this subject, Ifrah,[25][pp. 416-418, quoting a personal communication of Billard on an anonymous work published from Sholapur in 1962] argues at length to present an irrefutable proof of the existence of decimal place-value system with zero being used in harmony with it, in a 5th century CE Jaina work on cosmology, entitled *Lokavibhāga* (meaning, *The Parts of the Universe*). Interestingly, using astronomical information available in two particular verses (no. 52 and 53) of the text itself, one can decipher the exact date of its completion to be precisely on Monday, 25 August, 458 CE, whereby Ifrah concluded it to be "*the oldest known authentic Indian document to contain the use of zero and decimal numeration.*" However, this is not the end of the story. Translating the content of the verse no. 51, Ifrah points out that "*These could and very probably does mean that the current version of the Lokavibhāga is an exact* (Sanskritized) *reproduction of an original* (probably written in a Jaina dialect of *Prākṛt*) *which was written before 458 CE, by some Muni Sarvanandin*". But when did this *Muni* live? A hundred or a few hundred years back? With a touch of frustration, Ifrah comments "*We will never know.*" Perhaps posterity will!

Without making the list any longer[48] we conclude this section by quoting A.L. Basham's words of praise[4] for the unknown inventor of the decimal place-value system:

> "*Most of the great discoveries and inventions of which Europe is so proud.....would have been impossible if Europe had been shackled by the unwieldy system of Roman numerals. The unknown man who devised the new system was from the world's point of view, after the Buddha, the most important son of India. His achievement, though easily taken for granted, was the work of an analytic mind of first order.*"

7. GRADUAL GENESIS OF 'ZERO' FROM ŚŪNYA

The Sanskrit word "*śūnya*" was transliterated into Arabic in Al-Khwarizmi's[49] (780-850 CE) work De Numero Indorum[50] as '*sifr*', meaning 'empty'. The associated Arabic root √*sfr* leads to other words of similar meaning, like *asfara* (meaning '*empty*') or *safir* (meaning 'to be empty'). The slightly changed term '*sifra*' was used to denote '*zero*' in *Sefer ha misper* (meaning, 'Book of Numbers') by Rabbi Ben Ezra (1092-1167). This one, along with many local variants like *cifra*, *cyfra*, *cyphra*, *zyphra* etc., was used for the same purpose, i.e., to denote 'zero'. However, they took a turn by the beginning of the Renaissance and one may find the Spanish word '*cifra*', derived directly from Arabic '*sifr*' being used with a changed connotation to refer to the rest of the digit numbers. Similar examples prevail in the case of other European languages as well—for instance, '*chiffre*' in French, or '*ziffer*' in German are still used to refer to digit numbers. However, the Arabic sifr was gradually latinized as '*zephyrus*',—'the west wind' and Leonardo of Pisa, better known as Fibonacci, while writing his seminal book, *Liber Abaci* in 1202, coined the term 'zephirum' mere light breeze—almost nothing, to denote 'zero'. Later it got converted to zefiro in the Venetian dialect of early Renaissance, and our 'zero' is a mere contraction of that[51].

8. EARLY NON-MATHEMATICAL ROOTS OF ŚŪNYA

Since no written mathematical treatise of remote Indian antiquity is extant, historians have turned their attention to the non-mathematical texts of that period, with an intention to find tacit mathematical clues, if there be any. The idea is to dig deep into the academic ambience of that period and analyze in the light of those works, the available social contexts, towards a possible unearthing of some threads of mathematical thought, which might have been prevalent during that period. If non-mathematicians are found to be at ease with novel mathematical concepts, for example, that of 'zero' as a digit in its own right, in an arguably place value-system[52], it certainly strengthens the argument in favour of those peoples doing *Gaṇita* in that society to have already mastered those concepts. Two such cases are certainly

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worth mentioning, one directly referring to the 'zero' of mathematics and the other tacitly applying a similar concept.

8.1 Direct Evidence : Śūnya in Piṅgala's Chandaḥsūtra

The *Piṅgalacchandaḥsūtram* i.e., the rules of prosody for both Vedic and Classical Chandas given by *Piṅgala*[53] [also known as *Piṅgalācārya* (in *Halāyudha*), *Piṅgalmuni* (in *Mahābhārata*, where he is found to be the chief priest in the serpent sacrifice of King *Janamejaya* to avenge of his father's death[54]) or *Piṅgalanāga* (in *Jivānanda Vidyāsāgara*[63])] come into the limelight as an irrefutable landmark in the history of mathematics, particularly in connection with the priority in the inception of place-value notation, as in two of its *sūtras*, it formally mentions the term '*śūnya*' for the first time in the history of human civilization (as far as our present evidences stand), a clear mathematical concept of '*zero*' and discusses formulations for intricate combinatorial calculations towards what is now recognised as binary arithmetic, going up to conversion of a decimal number to binary and vice versa—an extraordinary feat indeed.[42]

Analyzing the reasons behind the abundance of metrical verse based resources of Indian Antiquity and the consequent indispensable need for careful and systematic analysis of poetic metrics, A.K. Dutta concludes "*Ancient Indians had the perception that the metrical form has greater durability, power, intensity and force than the unmetrical, and hence recorded all important knowledge in verse form. In particular, all ancient Indian mathematical literature, beginning with the pre-Pāṇinian śulba-sūtras, are composed entirely in verses. This tradition of composing terse śutras,*[55] *(often by avoiding the use of verbs as far as possible and compounding nouns at great length), which could easily be memorized, ensured that, some of the core knowledge got orally transmitted to successive generations.*"[16]

The Vedic seers attributed almost mystic significance to *chandas*[56]. Consequently, close attention was paid to the study of *chandaḥśāstra* i.e., the science of verse meters, along with its language, prosody and the proper time and place for its recitation. It is stated in the *Śruti* that, he who made rituals possible by the brahmins, should know the meter of each *mantra*. Along with the study of a *mantra*, one must have a clear knowledge about the deity, meter and the seer associated with it, failing which the rituals performed will not attain the desired goal.[56] Further in this regard, one may also quote from *Bṛhaddevatā*, [verse 136, VIII] supposedly written by the Vedic seer *Śaunaka*[61]:

aviditvā ṛṣiṃ chando daivataṁ yogameva ca
yo'dhyāpayejjapedvāpi pāpīyāñjāyate tu saḥ

One who teaches or recites [the mantras] without knowing
the seer, the meter, the God, and the proper application of it becomes indeed a sinner.

Hence it is no wonder that the study of analyzing the meters became a very respectable study right from the early Vedic society onwards. *Piṅgalachandaḥsūtram* is the only authentic extant complete work, though not the earliest, in this field[57]. The unknown author of *Agnipurāṇa* refers to and summarises the work of *Piṅgala.* He states:

> *chando vakṣye mūlajaistaiḥ*
> *piṅngaloktaṃ yathākramam*
>
> *I shall now state about the chandas*
> *in the order uttered by Piñgala, our Elder*

There exist many commentaries on *Piñgalachandaḥsūtram*, done at much later dates, the most widely mentioned ones being *Kedārabhaṭṭa's Vṛttaratnākara* (1150 CE), *Yādavaprakāśa's* commentary (circa 1050 CE) and *Mṛtasañjīvanī* by *Halāyudha* (11th century). Though there were many writers of metrical books earlier than Piṅgala, as is testified in his book itself,[58] unfortunately only the names of those authors are available, their works having presumably perished with time.

Now, what attracted the attention of the historians of mathematics, particularly those dealing with antiquity, towards the work of this master prosodist, *Piñgala*? Indeed, after a critical examination of all the 315 *sūtras* spread over the eight chapters of his work, one finds the *sūtras* 20-35 in the eighth chapter to be of immense mathematical potential. As usual, they are in keeping with the spirit of pithy and cryptic presentation of a '*sūtra*', almost obscure to a non-specialist. However, thanks to the later commentaries, they can be relatively easily deciphered to produce an excellent account of combinatorial calculation along with binary arithmetic.[42][61] We narrow our focus of attention to the *sūtras* 28-31, dealing with the combinatorial question: how many different meters are there with a given length? That is, to compute the total number of possible arrangements of long and short syllables (respectively, the *guru* and the *laghu*), repetition allowed, without actually constructing the arrangement of all possible combinations of guru and laghu in a given meter (called the *prastāra*, a bed or matrix in which the *gurus* and *laghus* are listed horizontally)[59]. Of particular interest for us in this article are the *sūtras* 29-30, *rūpe śūnyam* and *dviḥ śūnye* by name, clearly mentioning '*śūnya*'. However to see them work as mathematical '*śūnya*', one has to combine them along with the *sūtras* immediately preceding and immediately succeeding them, viz. *dvirardhe* and *tāvad ardhe tad guṇitam.* Four of them, put together in a row, and interpreted according to the commentators, give us the necessary rule of computation, that clearly involves a notion of mathematical "zero". The four *sūtras* are beautifully translated into English by Sarma [55] as follows[60]:

> *sūtra* **28** : "[First write down the number of syllables in the given meter and go on halving that number. Each time] **when** [the number is] **halved** (*ardhe*), [write down in a separate row or column the digit] **2** (*dviḥ*).
>
> *sūtra* **29** : "[When you reach an odd number, subtract 1 from it.] **Whenever** **1**[61] [is subtracted (*rūpe*), write down in a separate column a] **zero** (*śūnyam*).

sūtra **30** : "[Continue thus until the process stops. Then where you wrote a] **zero** (*śūnye*), [multiply by] 2 (*dviḥ*).

sūtra **31** : "**Where** [the number was] **halved** (*tāvad ardhe*), multiply [the result of the second process] by itself (*tad guṇitam*)."

Following the above rule, we may construct the following concrete example, pertaining to the *uṣṇik* meter having total twentyeight syllables, whence each *pāda* in its *samavṛtta catuṣpāda* variation has seven syllables. We are to find the number of all possible arrangements when each one of the seven places may be occupied by either a *guru* or a *laghu*. An elementary knowledge of combinatorics tells us that the answer is 2^7.. In the table below, let us verify this following *Piṅgala*. Columns under A and B refer to the *sūtras* 28 and 29, whereas the column under C shows the calculation done from the bottom towards the top according to the *sūtras* 30 and 31. Note that, while beginning the calculation in column C, *one has to start taking 1 by default*, and then apply the rules 30 and 31 as the case may be, a point worth mentioning, as it is not clearly stated in the (translation of) *sūtra* 30.

uṣṇik (*samavṛtta*; *catuṣpāda*)	A	B	C
1. Write the number of syllables	7		
2. 7 cannot be halved; so reduce it by 1	6	0	$(2^2 \cdot 2)^2 \cdot 2 = 2^7$
3. Halve 6	3	2	$(2^2 \cdot 2)^2$
4. 3 cannot be halved; so reduce it by 1	2	0	$2^2 \cdot 2$
5. Halve 2	1	2	2^2
6. 1 cannot be halved; so reduce it by 1	0	0	$1 \cdot 2$
	Stop.		Begin upwards

So it appears that *Piṅgala* prescribed the use of two symbols, viz. that of two and zero,[62] towards distinguishing between two kinds of operations, squaring and multiplying by 2. Surely, such a calculation could have been done with any two arbitrary symbols. But, his choice favouring the term '*śūnyam*' clearly indicates that during his time, a concept of mathematical *śūnya* was prevalent. But what kind of *śūnya* was it? Did it only signify absence of an operation, in a spirit similar to the *Pāṇinian* concept of *lopa*, or that of a decimal place-value system in vogue? Van Nooten[42] observes that, "*we can be reasonably certain that his counting system was predicated on a base of ten.*" He further points out that a system of notation that simply uses two symbols as markers of a place value does not necessarily produce a binary system. But if there is a system that uses two symbols in such a way that every string made with them has a unique decimal equivalent and it is shown how one may convert a decimal number into a string of those two symbols uniquely, "*then indeed we do have a binary system. Piṅgala has done at least this much.*" In the view of Needham[40], "*Place value could and did exist without any symbol for zero. ...But zero symbol as a part of the numerical system never existed* *and could not have come into being without place-value.*" Along this line of argument Sarma[55] concludes "*...the invention of decimal place-value system along with the concept and symbol of zero must antedate considerably* Piṅgala's *mention of the zero symbol.*"

As mentioned earlier, the time of the original composition of *Piṅgalachandaḥ-sūtram* is hotly debated among the scholars. But setting his time frame accurately, as far as possible from the circumstantial evidences, is extremely crucial, as this may pave the path towards a rejection of the so-called Eurocentric view of Greek origin of the concept of zero and its being transmitted to the Indian soil through the Alexandrian invasion of the northern India (326 BCE)[59](pp.63)[34](pp. 37). In the resulting tug-of-war, we find Sastri[56](pp.81) placing him in 4th century BCE while Dutta and Singh[9] and Dowson[14] argue in favour of placing him around at least 200 BCE, if not earlier. Again, S.R. Sarma[54] is in favour of keeping the time-frame a bit wider, between 400 to 200 BCE, which is contested by Bronkhorst[7], who not only claims that the period will be much later but also suggests that the eighth chapter of the book, where the clear mention of *Śūnya* can be found, is a later work, not original to *Piṅgala*. On the other hand, Van Nooten[42] in his paper, while attributing the credit of invention of binary numbers to Piṅgala, recognizes the eighth chapter as original[63], but analyzes the time-frame critically and is not ready to put him before 3rd century of CE. In his opinion, since the earliest known reference to *Piṅgala* is found in *Mīmāṁsāsūtra* of *Śabara*, whose time has been assigned to 4th century CE, and as "*the internal evidence of the treatise does not militate against a date before the 2nd century A.D. since the composition of the sūtras follows the pattern of the older unversified sūtras, such as those contained in Pāṇini's Aṣṭādhyāyī*", "*it is not possible, on objective grounds, to decide whether Piṅgala's treatise preceded or followed Pāṇini, nor is it possible to prove that Piṅgala's work existed before the third century A.D.*" But Sarma in a comparatively recent article[55] strongly refutes this claim and puts forward fresh arguments and views to the effect that the eighth chapter is indeed original to *Piṅgala*[64] and "*there are close similarities in the methods of exposition and notation between the Chandaḥsūtra on the one hand and the Aṣṭādhyāyī and the Vedāṅgajyotiṣa on the other. The affinity with these two texts places the Chandaḥsūtra at about 400 BCE.*"

All these show the importance of a comparative time-frame based study of cross-references available on the two literary giants of Indian Antiquity, *Pāṅini* and *Piṅgala*. One can find a point by point comparison in A. Sastri[56], although some experts don't find it very convincing:

> *Pāṅini* dealt with *sūtras* both *laukika* and *vaidika*; *Piṅgala* dealt with chandas both classical and Vedic.
>
> *Pāṅini* had recourse to *pratyāhāras* to make his sūtras concise; *Piṅgala* reflects a similar spirit introducing the *gaṇas* in analyzing *chandas*.
>
> *Pāṅini* begins with *vṛddhirādaic* as an auspicious *sūtra*; *Piṅgala* took a similar stance with the *dhīśrīstrīm* which suggests auspiciousness.

Pāṅini at first has given the definition of lengthening; *Piṅgala* of the lengthened syllables.

Pāṅini used the word *vṛddhi* as *nāndī*; *Piṅgala* in his turn used *dhī*, a term depicting practically the same connotation.

The first verse of *Pāṅiniya* śikṣa runs as follows:

atha śikṣāṁ pravakṣyāmi
pāṇinīyaṃ mataṃ yathā

I now talk about phonetics, according to the doctrine of Paṇini.

The commentator of this verse interestingly remarks[56] (pp. 76)

vyākhyāya piṅgalācāryasūtrāṇyādau yathā yathā
śikṣāṃ tadīyāṃ vyākhyāsye pāṇinīyānusāriṇīṃ

Upon having explained the rules of Piṅgalācārya as it is,
I shall now explain his another work śikṣa following Pāṇini.

The *Śikṣāprakāśa* states that *Piṅgalācārya*, the younger brother, carefully followed the direction of *Pāṇini*, his elder brother^{65}, who had probably initiated him into the study of metrical science.[56] Also *Ṣadguruśiṣya* in his commentary (1187 CE) on *Ṛganukramaṇī* refers to *Piṅgala* as a *Pāṇinīyānuja*[61] (pp.40), the younger brother (i.e., *anuja*) of *Pāṇini*. Interpretating 'younger brother' in the sense of being junior in the same intellectual lineage, one can now claim to have enough reasons to consider *Piṅgala* to be a younger contemporary of *Pāṇini*. But when did *Pāṇini* live? K.V. Sarma[52] settles for 600 BCE. Pandit[44] in his groundbreaking work has put him somewhere between 700-400 BCE. Thus going by the majority view, we could safely place *Piṅgala* before the third century BCE.

8.2 Indirect evidence : Tacit concept of 'zero' in Pāṇini

Pāṇini, the great grammarian, the author of *Aṣṭādhyāyī*, an archetypal work, thoroughly systematizing the Sanskrit grammar, which is studied by the linguists even today as the 'work of a linguist and not a language-teacher', exhibited extraordinary technical and descriptive skills. Irrespective of the controversy on whether there existed earlier grammatical schools in Sanskrit, or how many grammarians actually predated *Pāṇini*, there seems to be no doubt among the scholars that this peerless work stands alone, 'a literary monument', among all the pre- or post-*Pāṇinian* grammars on Sanskrit, known to us. Out of a plethora of commentaries available on his work, two particularly important are by *Kātyāyana* and *Patañjali*, who regarded this work as a 'scientific system based on certain scientific principles' and tried to establish it through their original contributions towards the proper interpretation and understanding of the original work. Passion for attaining perfection in grammar, recognized as a *Vedāṅga* as we have already pointed out, was of paramount importance in the Vedic studies in ancient India.

It is from this background, researches deep into the analysis of the *Pāṇinian* grammar, which has an inherent scientific 'structure', have been carried out by scholars like Allen[1], Hass[22], Bloomfield[6] and S. Al George[17] during the middle of 20th Century CE. In the words of Bloomfield[6], '*The Hindus hit upon the apparently artificial but in practice eminently serviceable device of speaking of a zero element.*' In a later elaborate work of truly seminal nature, Pandit[44] comments "*The technique of zero is thus basically and purely a technical device born out of necessity for brevity and symmetry of linguistic description........the only technical principle that Pāṇini might have and has actually used is the principle of zero, perhaps borrowed from positional mathematics.*" Of course, there is no way one can be more categorical about this possible correlation. Indeed, A.K. Dutta[16] comments: "*One does not know whether the mathematical zero existed by his time and whether he was influenced by it, or whether it was Pāṇini's grammar which inspired the great mathematical invention.*" But *Pāṇini techniques of linguistic analysis do consist of some basic fibers of abstract universality, making one feel that they "are comparable to those adopted in mathematics, computers, symbolic logic and artificial intelligence.*"[20]

Pāṇini has employed three techniques named *lopa, it-samjñā* and *pratyāhāra* for grammatical description and one called *anuvṛtti* pertaining to the interpretation of *sūtras*. Pandit argued at length to establish that *all these four seem to have been based on one single principle,* viz. *'lopa', which can be rendered by the modern term 'zeroing'*. These techniques by their nature are very complicated and too technical for the comprehension of those not well-versed in Sanskrit grammar, and in any case happen to be way beyond the field of specialization of the present author. *Pāṇini*'s use of his grammatical or linguistic zero '*lopa*'is used as a marker of an empty (*śūnya*) non-occupied space or position. His relevant basic *sūtra* is : *adarśanaṃ lopaḥ*, found in *Aṣṭādhyāyī*, (I.I.60) which means[66] (non-appearance [of a sound or morpheme[67]] is *lopaḥ*).[68] That the *Pāṇinian* '*lopaḥ*' stood for a sort of 'zero' is also indicated by the fact that *Pūjyapāda* (c. 450 CE) in his *Jainendra Vyākaraṇa* replaced 'lopaḥ' in the *sūtra* VIII.4.64 with '*khaṃ*', which was a common term for zero.

For the purpose of linguistic description of the Sanskrit language of his time, *Pāṇini* in his grammar tried to formulate a rigorous logical structure from which the whole body of the Sanskrit language might be seen to evolve. Towards this, he framed his *sūtras*, minimum in number, each one encompassing as many cases as possible. Dealing with the verbal formation of an existing Sanskrit word, he reconstructed the evolution of it from its root according to his perception of symmetry in description and while doing this analysis, he often encountered a situation where there was neither a phoneme nor a morpheme, but to match the structure, as proposed by him, he had to account for something. *Pāṇini* proposed the concept of 'zero' (called *lopa* in his terminology) to save the situation, and one may find abundant use of this concept in his work. The following example is taken from Pandit[44]. Taking the word daśati from spoken language, *Pāṇini* reconstructed its root as √*daṁś*, 'to bite' and observed that its present tense, third person, singular number form should have been :

daṁś + (*ś*)*a*(*p*) + *ti* = *daṁśati*, where the nasal ṁ ought to be present. But it was spoken as daśati. Hence *Pāṇini* had to make the provision for '*lopa*' of the nasal in this case. He did it by framing the rule 6.4.25, *daṁśasañjasvañjām śapi*—i.e., the nasal of roots √*daṁś*, √*sañj* and √*svañj* disappear before *śap*, the *vikarana* (i.e., the middle morpheme) of the first conjugation, symbolically :

da(ṁ)śati = da(0)śati = daśati.

Thus the nasal, which was due in this case (*prasakta* i.e., attached), was not to be pronounced, according to the *Pāṇinian* reconstruction.[69] Pandit submits that this use of *lopa* is much the same, structurally, as the use of 'zero' as placeholder in mathematics, while it is used in writing a numeral, say 202, to indicate the vacant place.[70]

Accepting the possibility that "*the great grammarian Pāṇini seems to have some clues for the concept of the mathematical zero*", Paramhansa[46] sceptically submits that "*the zero in Pāṇini has nothing to do with the mathematical zero of the place-value notation. A caution for the* '*lopa*' zero *for Pāṇini* .... *It is just subjective zero as relative zero of Abstract Algebra. For example, a non-trivial normal subgroup H of a group G is the 'zero' of the quotient group G/H.*" He further gives linguistic examples in favour of his argument. However, Pandit has categorically established with several examples[45] that the *Pāṇinian* technique of zero used in linguistic descriptions is comparable to the mathematical zero as a place holder, a marker for the vacant place, much like the zero in numbers 10, 20, 30 etc. To conclude this section, we quote from Pandit[45] some salient similarities of the two concepts[71], 'zero in Mathematics' and '*lopa* in *Pāṇini*' :

(i) *in both of them, the concept of zero attains the status of a technical device to build up the respective systems*;

(ii) *in both of them, zero represents the zeroed entity as well as its place*;

(iii) *both presuppose positional analysis and representation*;

(iv) *in both of them, the concept of zero is not merely intellectual but can be comprehended, arrived at and represented by a definite empirical, formal process laid down in those sciences*;

(v) *in both of them, the zero requires to be inserted at the deep-structure level.*

9. GREEK CONNECTION?

As we have already pointed out, there are a school of historians of Mathematics, according to whom, the 'decimal place-value principle with zero as a hall-mark', was in actuality not an independent Indian achievement. The root of it presumably lies with the Babylonians (Akkadians), who by 300 BCE had started using a queer symbol of two slanted wedges (mostly, with certain exceptions as well[34]) to denote an empty place-mark in a written numeral on unbaked clay tablets, instead of keeping a 'blank space', as they had been doing before, for over a thousand years. Though these historians seem to give some concession to the decimal place-value system in favour of India, which according to them was being employed in India, especially among the Jainas and Buddhists, towards the beginning of the common era' (i.e., Christian era), as far as the inception of '*zero and its symbol*' is concerned, they opine to the contrary. Laying an undue stress on the 'symbol' part of it, they claim that '*Greek (astronomical) Papyri of the period immediately preceding and following the beginning of the common era demonstrate that, they filled (the blank space) with an adaptation of the Akkadian symbol for zero (two slanted wedges); this adaptation looks like a circle with a bar over it* (Ō)[72]*. This, then, is the origin of the form of the symbol for zero*[73] *as a circle.*' Further arguments amount to the claim that, during this period, the Greeks came to India in large numbers, brought with them documents with such a symbol written there and while translating them into Sanskrit (with which the Indian mathematicians/astronomers like *Varāhamihira* are credited) '*the symbol for zero* (i.e.,Ō) is taken as a circle *(pūrṇa*, the full moon) or a dot (*bindu*).'[49] The earliest such incident, according to D. Pingree, took place around 425 CE, the principal impetus behind the whole theory being '*there is no certain evidence that a symbol for zero was in place* (*in India*) before the 5th century A.D.'[74] However, in the context of what we have discussed so far in the previous sections, we find such interpretations by Pingree and the co-sharers of his opinion untenable. It appears more probable that Indians, even in pre-common era, were familiar with the mathematical perspectives of *Sūnya*. Whatever symbol for this concept they might have used at that early stage, is of course not known to us.

10. EPILOGUE

Historical reconstruction of mathematical knowledge, that was likely to be prevalent in Indian Antiquity, is much like arranging an enormous zig-saw puzzle, many pieces of which are missing.[75] Competent historians of Mathematics, all over the world, are trying to arrange the available pieces, according to their own respective 'stance's, with an obvious intention to try and guess the picture it may suggest. And the job is anything but easy. Patriotic passion or pre-conceived ideas[76] often come in the way of scholarly acumen, prompting one to misplace one or two pieces or perhaps not to place them at all, distorting the figure to accommodate one's 'stance'. Some are dedicated to make it look like the dancing Śiva, while few others seem to be determined to find a dancing monkey![77] Of course, there is a third group, carefully steering a discreet middle course, analysing objectively as far as possible, the available pieces of evidence or information and sometimes even the lack of information, trying to make a pattern out of it. With every new piece being found occasionally, as a new input to the zig-saw puzzle, one has to try and find its right place in the puzzle, sometimes destroying the existing pattern. And the journey continues. We put an end to our discussion by slightly altering what Bourbaki[78] once said about mathematicians :

Historians of Mathematics have been correcting their errors to the consequent enrichment *and not impoverishment of this discipline; and this gives them the right to face the future with serenity.*

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REFERENCES

1. W.S. Allen; *Zero and Pāṇini*, Indian Linguistics, Vol. 16, (Nov. 1955) pp. 106-113
2. T.A. Sarasvati Amma; *Geometry in Ancient and Medieval India*, Motilal Banarsidas, Delhi, (1999)
3. Sri Aurobindo; *The Foundation of Indian Culture*, Sri Aurobindo Ashram,Pondicherry.
4. A.L. Basham; *The Wonder That was India*, pp. 496.
5. Roger Billard; *L'astronomie indienne*, Ecole Française d'Extreme-Orient, Paris, (1971)
6. L. Bloomfield; *Language*, 1961, pp. 132
7. J. Bronkhorst; *A Note on Zero and the Numerical place value System in Ancient India*, Asiatische Studien/Études Asiatique, 48.4, 1039-42 (1994)
8. H.N. Chakravarty; *The Concept of Śūnya in Tantra and Āgama*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) p. 197
9. B.B. Datta and A.N. Singh; *History of Indian Mathematics*, 2 Vols. Asian Publishing House (1935)
10. B.B. Datta; *Early Literary Evidence of the use of the Zero in India*, Amer. Math. Monthly 33(Nov. 1926) pp. 449-454
11. B.B. Datta; *Early Literary Evidence of the use of the Zero in India*, Amer. Math. Monthly 33(Dec. 1926) pp. 566-572
12. R.C. Datta; *Ṛgveda Saṁhitā*, Haraf Prakasani, Calcutta, (1976) pp. 132, 292.
13. Leo Depuydt; *Review of the Book: Unexpected Links between Egyptian and Babylonian Mathematics by J. Friberg,The Mathematical Intelligencer*, vol. 30, No. 3, 2008; pp.72
14. John Dowson; *A classical dictinoray of Hindu Mythology and religion*, p. 11

15. Antonio J. Durán(edt.), *The Life of Numbers*, Biblioteca Nacional, Ministerio De Cultura, Spain, (2006)

16. Amartya Kumar Dutta; *Mathematics in Ancient India* : Some Glimpses, Bharatatatta, Ramakrishna Mission Institute of Culture, Kolkata, India (2009)

17. S. Al George; *The Semiosis of Zero according to Pāṇini*, East And West, 17 Nos. 1-2, Rome, 1967, pp. 116-119

18. R.C. Gupta; *World's longest lists of decuple terms*, Gaṇita Bhāratī, vol. 23 (2001), pp. 83- 90.

19. R.C. Gupta; T*echnology of Using Śūnya in India*, The Concept of Sunya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, pp. 24 (2003)

20. R.C. Gupta; *Some new studies and findings regarding ancient Indian mathematics, Mathematics Teacher*, vol. 38 (2002), pp. 95-107.

21. G.B. Halsted; *On the Foundation and techniques of Arithmetic*, pp. 20

22. W. Hass; *Zero in Linguistic Description*, Studies in Linguistic Analysis, Special Volume of Philosophical Society, Oxford, 1957, pp. 33-53

23. Takao Hayashi; *Bakṣālī Manuscript,Encyclopaedia of Classical Indian Sciences*, H. Selin, R. Narasimha (edt.), Universities Press, (2007) pp. 52-55

24. A.F.R. Hoernle; *The Bakshālī Manuscript*, The Indian Antiquity 17, (1988) pp. 33-48 and 275-279

25. Georges Ifrah; *The Universal History of Numbers*, John Weily and Sons (2000)

26. Swami Jitatmananda; *Śūnya* : *Its meaning in Philosophy and Science*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 1-18

27. A. Jones; Astronomical Papyri from Oxyrhynchus, American Philosophical Society, (1999)

28. G.G. Joseph; *The Enormity of Zero, Mathematics Teacher*, Ann. Conf. of AMTI, Cochin, (2001) pp. 108-125

29. G.G. Joseph; *The Crest of the Peacock—Non European Roots of Mathematics*, Penguin Books, (1992) pp. 108-125

30. S.C. Kak; Astronomy of Vedic Altars; Vistas in Astronomy, 36, (1993) pp. 117-40

31. S.C. Kak; The sign for zero; The Mankind Quarterly, 30(3) (1990) pp. 199-204

32. S.C. Kak; Indus Writing; The Mankind Quarterly, 30 (1989) pp. 113-118

33. S.C. Kak; Indus and Brahmi, Further Connections; Cryptologica, 14 (1990) pp. 169-183

34. Robert Kaplan; *The Nothing That Is : A Natural History of Zero*, Oxford University Press, (2000).

35. F.E.J. Linton; *Shedding Some Localic and Linguistic Light on the Tetralemma Conundrums*, Contributions to the History of Indian Mathematics, Emch, Sridharan, Srinivas (edt.), Hindusthan Book Agency, (2005) pp.63-73

36. R.N. Mukherjee; *Zero*, Encyclopaedia of Classical Indian Sciences, H. Selin, R. Narasimha (edt.), Universities Press, (2007) pp. 441-444

37. Rabindranath Mukherjee; *Discovery of Zero and its impact on Indian Mathematics*, Dasgupta & Co. Pvt. Ltd., Calcutta (1991)

38. A. Mukhopadhyay; *Impact of Indian Philosophy on Bhāskarīya Concept of Zero*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 197

39. V.K. Murty; *A Brief History of Indian Mathematics, Prabuddha Bhārata*, (Sep. 2007) pp. 517-523,

40. J. Needham and Wang Ling; *Science and Civilization in China*, Vol. II, Cambridge, (1959) p. 10

41. O. Neugebauer; *The exact Sciences in Anitiquity*, Princeton, (1952)

42. B. Van Nooten; *Binary Numbers in Indian Antiquity, Journal of Indian Philosophy*, 21, pp. 31-50, (1993)

43. J.J. O'Connor and E.F. Robertson; An Overview of Indian Mathematics, http://www history.mcs.st-andrews.ac.uk/HistTopics/Indian_mathematics.html (November, 2000)

44. M.D. Pandit; *Zero in Pāṇini*, University of Poona, (1990).

45. M.D. Pandit; *Reflections on Pāṇinian Zero*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp.101-125

46. S.A. Paramhans; *Philosophical and Mathematical Implications of Zero in Indian Culture*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 206-210

47. Ian Pearce; *Indian Mathematics: Redressing the Balance*, http://www-history. mcs. standrews. ac.uk/history/Projects/Pearce/index.html (May, 2002)

48. Kim Plofker; *Mathematics in India*; Princeton University Press, (2009)

49. David Pingree; *Zero and the Symbol for zero in early sexagesimal and Decimal Place-Value Systems*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp.137-141

50. C.K. Raju; *The mathematical Epistemology of Śūnya*, The Concept of Śūunya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 170-183

51. P. Ray and S.N. Sen(edt.); *The Cultural Heritage of India*, Vol. VI (Science and Technology), The R.K. Mission Institute of Culture, Kolkata (1986).

52. K.V. Sarma; *Word and Alphabetic numeral Systems in India*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 59

53. S.A.S: Sarma; *Vedic numerical System including Śūnya*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 32

54. S.R. Sarma; *Śūnya, mathematical Aspect, Kalātattvakośa* : A Lexicon of Fundamental Concepts of the Indian Arts, Bettina Baumer (edt.) Vol. 2, pp. 400-411, (1992)

55. S.R. Sarma; *Śūnya in Piṅgala's Chandaḥsūtra*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp.126-136

56. Asoke Chatterjee Sastri; *Piṅgalacchandaḥsūtra* : A study, University of Calcutta, (1987)

57. R. Sathyanarayana; *Śūnya : A Holistic View*, The Concept of Śūnya, A.K. Bag and S.R. Sarma (edt.), Aryan Books International, (2003) pp. 264-287

58. A. Seidenberg; *Origin of Mathematics, Archiive for Hist. of Exact Sciences*, 18(4) (1978), pp. 301-342

59. Charles Seife; *Zero—The Biography of a Dangerous Idea*, Penguin Books, (2000)

60. R.S. Sarma (edt.); *Brāhmasphuṭa Siddhānta*, Indian Institute of Astronomical and Sanskrit Research, New Delhi (1966) pp. 243

61. R. Sridharan; *Sanskrit Prosody, Piṅgala Sūtras and Binary Arithmetic*, Contributions to the History of Indian Mathematics, Emch,Sridharan,Srinivas(edt.), Hindustan Book Agency, (2005) pp. 33-62

62. Frits Staal; *On the Origins of Zero*, Studies in the History of Indian Mathematics, C. S. Seshadri (edt.), Hindustan Book Agency, (2010) pp. 39-53

63. *Jīvānanda Vidyāsāgara*; *Pingalacchandaḥsūtra* with the *Mṛtasañjīvanī* commentary by *Halāyudha* and a further commentary, Calcutta, (1928)

NOTES

1. The mathematical heritage of the Indian subcontinent has long been recognized to be extremely rich. Hundreds of thousands of manuscripts in India and elsewhere, mostly written in Sanskrit, attest to this tradition.[48]

2. Otto Neugebauer (Zur Entstehung des Sexagesimalsystems, Göttingen, 1927) has shown such examples in an astronomical tablet of Selucid period of Babylon. There is a hypothesis that this might have been a copy of some earlier documents (H. Hunger, Spätbabylonische Texte aus Uruk, Berlin, 1976). But these are only suppositions. Only further archaeological discoveries can provide definite proof.[25]

3. "Sedulo curavi, humanas actiones non redere, non lugere, neque detestari, sed intelligere." The translation given here, is credited to Sridharan[61].

4. Quoting R.C. Gupta[20],"in respect of Vedas and Vedic Corpus, there seems to be no moderation in regard to highly fantastic claims of advanced scientific achievements. These include the proof of famous Fermat's last equation and the Einsteinian mass energy equation...."

5. Roger Billard[5] in 1971 demonstrated in his book that Āryabhaṭa has made his own independent observations and his astronomical constants and parameters are not borrowed from the Greeks.

6. Plofker concludes that "*much work still needs to do done before the issue can be confidently resolved, if it ever can be*" and finally she expresses her preference to the transmission hypothesis

7. A good example may be the uncertainty of time period regarding the inception and development of Vedic religion. Following remark by Pearce in [47] is interesting: "*S. Kak[30]*, a noted authority on Vedic Astronomy, states in a recent work that the time period for the *Vedic religion stretches back potentially as far as 8000 BC and definitely 4000 BC. Whereas G. Joseph states 1500 BC as the forming of the Hindu civilization........R. Gupta in his paper on the problem of ancient Indian chronology shows that dates from 26000-200 BC have been suggested for the Vedic period. Having consulted many sources I am confident at placing the period of the Vedas* (*and Vedangas*) *at around 1900-1000 BC.*" *Again, Mukherjee* [37] *points out that, Sampurnananda in his book Āryoṅ kā Ādideśa* (*in Hindi*) *has argued Ṛgveda to be 17000 years old.*

8. It is almost certain that when the Aryans arrived in India they brought no form of writing with them.....Their intellectual and spiritual leaders would certainly have had a "knowledge of the great religious poems; but it seems that their literature was written at a later date, and then the literate men would doubtless have preferred to keep the oral tradition going as long as possible to perpetuate their prestige and their privilege"[M Cohen, 1958][25]

9. both the words come from the same Sanskrit root √vid, that means 'to know'

10. The approximate dates mentioned in this paragraph are taken from Joseph[29], who however puts in clear terms the state of affairs regarding the uncertainty in predicting the time period of early Indian texts. "*The dates given here are rough and conservative estimates of when the first versions of the texts were recorded. It is very likely that before they were written, an earlier oral tradition kept the contents alive. Copying old texts was a common pursuit of the Indian scholar and student, sanctioned by religion and custom. It is therefore important not to depend on evidence of the date of copies of mathematical texts in assessing the true age of a particular method or technique.*"

11. In Muṇḍakopaniṣad, one may find [61] the following verse, mentioning the six limbs of the Vedas : *tatrāparā ṛgvedo yajurvedas sāmavedo 'atharvavedaḥ śikṣā kalpo vyākaraṇaṃ niruktaṃ chando jyotiṣamiti.*

12. The word sulba is derived from the root √*śulb*, meaning '*to measure*', whence the term *śulba* stands for measuring cord.

13. One can refer to the groundbreaking work of Sarasvati Amma[2] in this regard. A. Seidenberg[58] pointed out that *Taittiriya Saṃhitā* contains the algebraic, geometric and computational aspects of the so-called *Pythagoras theorem.*

14. It is suggested that they were Vedic priests and skilled craftsmen.[43]

15. However, Sarasvati Amma places it between 5th to 8th century BCE.[2]

16. In *Viṣṇudharmottara purāṇa*, one finds an apt definition:

alpākṣaram asandigdhaṃ sāravat viśvatomukham
astobham anavadyaṇ ca sūtraṃ sūtravido viduḥ.

It is translated by Sridharan[61] as "A *sutra* should have the least number of syllables, should contain no doubtful words, no redundancy of words, should have unrestricted validity, should contain no meaningless words and should be faultless"

17. For a comprehensive list of various terms equivalent to *śūnya* used to denote 'zero' in Sanskrit literature, one may refer to Gupta[19] or Sarma[52]. An exhaustive dictionary of the terms pertaining to the Indian numerical symbols in antiquity, along with etymological notes for each of them, may be found in Ifrah [25], pp. 445-510.

18. Typical examples are the invocations (*śāntipāṭha*) of the *Iṣopaniṣad* and *Brihadārnyaka Upaniṣad*:

pūrṇam adaḥ pūrṇam idam pūrṇat pūrṇam udacyate
pūrṇasya pūrṇam ādāya pūrṇam eva avaśiṣyate

The infinite fullness is that Brahman. The same infinite fullness is this Brahman
The same infinite fullness springs from (is hidden) in the finite universe. When this same infinite fullness is taken away from the infinite fullness, still what remains is infinite fullness.[26]

Gupta[19] observes that, mathematically the second half of this invocation implies $x - x = 0$, a statement which is satisfied by $x = 0$ as well.

19. Physicist Wheeler says that "Empty space is the place of most violent Physics."[26]

20. *Kaṭhopaniṣad 1.2.20* [53]

21. *Amarakoṣa*, a Sanskrit lexicon written by *Amarasiṁha* around 5th century CE, interprets it as *śūnyṃ tu vaśikaṃ tucchariktake*, which means, zero is also called vaśikaṃ i.e., void or empty, *tuccha* i.e, trifling, *riktaka* i.e., emptied or devoid of. [38]

22. In *Ṛgveda Saṁhitā* the word *śūnam* (3.33.13) is found[12], and though the word finally means a swollen state or hollowness, it has been taken to imply lack, want or adherence [38]

23. *nāsad āsīn no sad āsīt tadānīṃ nāsīd rajo no vyomā paro yat* meaning,'*Existence was not there, nor non-existence. The world was not, the sky beyond was neither.*'[26]

24. *Niṣkala Śiva*

25. He used a form of logic, *catuṣkoṭi*, the famous *tetralemma*[35], claiming the four states "*śūnya, non-śūnya*, both or neither" to be considered as both mutually exclusive and jointly exhaustive.

26. refer to section 8.1

27. It is found in *Pañcasiddhāntikā* of *Varāhamihira* (505 CE), though by name, not by symbols[10]. An earlier reference, in language, as *ṣaṭ binduyutāni*, and also as *ṣaṭ khayutāni*, meaning, 'six with zero', representing 60, (Incidentally, the most common parlance for mathematical *śūnya* in ancient India was *kha*) can be seen in *Yavanajātaka* (Horoscopy of the Greeks) of *Sphujidhvaja* (270 CE), who, according to his claim in the text, versified an earlier prose version of *Yavaneśvara* (150 CE), a great exponent of Greek astrology. However, this example is cited by Kaplan[34] in favour of his argument that '*...the hollow circle of kha and the solid dot of bindu came to India from Greece*'.

28. Translation as in [28]

29. It features at the beginning and at the end of a series of '*mantras*', worshiping the numbers as: *kṣudrebhya svāhā, paryaikebhya svāhā, prathamebhya svāhā* and so on upto *viṃśati svāhā*, then *pṛthaksahasrābhyā svāhā* and afterwards again *kṣudrebhya svāhā*. The initial *kṣuda* stands for a symbol of 'absence' (a tacit form of 'zero'), and the final one represents 'fullness'. The term *paryaikebhya* refers to the seriality among the occurrence of numbers. [37]

30. Also a Kashmirian version of *Atharvaveda* is known, where many examples of zero being represented by dot, both in the marginal notes and in the text itself, are found. While the numbers in the marginal note may be accounted for as later inclusion, "*the symbols in the text itself must be from the time of Atharvaveda (500 BCE)*", observes Mukherjee[36], a minority view, of course.

31. *chidraṃ kham ity utkam* ; *Gopatha Brāhman* 2.2.5[44]

32. It is preserved in the Bodleian Library at Oxford University.

33. However, one scholar interprets one particular dot among them to be actually a small circle[36]

34. The colophon to the section on *trairāśika*, 'the rule of three' reads: *This has been written by the son of Chajaka, a brāhmaṇa and king of mathematicians, for the sake of Husika, son of Vaśiṣṭha, in order that it may be used (also) by his descendants.*[23]

35. G.R. Kaye put it in 12th century CE,(however his view is now rated by many scholars as biased, Joseph[28] has criticized him in very strong words : *it is particularly unfortunate that Kaye is still quoted as an authority on Indian mathematics*) Hoernle[24] assigned 3rd to 4th century CE, B.B. Datta[11] gingerly at early centuries of CE, L. Gurjar places it in between 2nd century BCE to 2nd century CE, A.A.K. Ayyangar in 8th to 9th century CE, I. Pearce[47] in sometime before 450 CE, (with a hint that the current version is a copy of an earlier work), whereas R. C. Gupta[20], Kaplan[34] and Hayashi[23] in 7th century CE.

36. The inscription of Gwalior referred to here is by no means the oldest inscription, if one includes the charters inscribed on copper plates as well. At the beginning of 20th century CE, some scholars, mainly G.R. Kaye (who is by now well known for his Eurocentric bias) and some others who share his views, questioned the authenticity of these plates, claiming them to be possible later forgeries, without much of concrete evidence in favour of such a claim. Towards trying to establish the Greek priority as the real inventor of our numeral system, their prime objective was to establish that the place value system with a clear use of zero in it, could not have been there in India much before the second half of 9th century CE. However their viewpoint has been scrutinized closely and categorically rejected on the whole by many scholars, historians and Indologists[47],[28]. We only quote Ifrah[25] : "... *thus we can see that these authors had worked out their conclusions far better than their arguments.*" If we focus our attention on the inscription of the numeral 'zero' only, there are several unquestionably authentic documents (stone inscriptions) found in various places of Southeast Asia, places which came under the cultural dominance of India at that time. The earliest known such epigraphic documents date back to 683 CE, one being the Khmer inscription of Trapeang depicting the numeral 605 and the other one is the Malay inscription at Palembeng (Sumatra) depicting 606.[25]

37. This place is about 300 km. south of Delhi, in present day India. The inscriptions are from King Bhojadeva's reign and were found in a temple of Lord Visnu inside the Gwalior fort.

38. Written in Sanskrit, using *Nāgarī* numerals, the first one contains, among other numerals, the numbers 10 and 20 (as the number of corresponding stanza of the inscription), while the second one shows the numbers 270 and 187 in the context of the area of the piece of a land being offered to the God, to be converted into a garden of flowers, so as to offer 50 garlands of flower for daily worship.

39. He was honoured with the appellation *Ganakacakracūdāmani*, 'crowning jewel among the circle of mathematicians' by Bhaskara II, an eminent mathematician of later period (b. 1114 CE). However, his attempt to define division by zero as *khacheda* (*'cipher divided by cipher is naught'*) is not accurate. Modern Mathematics rules out 'division by zero'. Raju[50] opines to the contrary, a view worth mentioning. He puts forward a completely different perspective of śūnya, invoking 'infinitesimals' of non-standard analysis introduced by Abhraham Robinson in 1960.

40. For example, if zero is put at the right side of a number written in a place-value system, say binary (i.e., base 2), ternary (i.e, base 3) or, in general *n*-ary (i.e, base n), the original number gets multiplied by its 'base' number n. It is the lack of this characteristic feature, that compels us to ignore the Mayan claim over the inception of 'true zero' in an otherwise fascinating place-value based intricate system derived by them. The Mayans lived in the isolation of Yucatan peninsula and reached their peak during 250 BCE–900 AD. They used a sea-shell like symbol, along with nine more pictorial glyphs to represent their 'zero' in different contexts[25][pp. 321]. How this civilization, with a definite passion for arithmetical calculation, almost amounting to an obsession, suddenly vanished into thin air, is a mystery yet to be solved.

41. For the actual verse along with its translation in English, see Dutta[16] See also pp. 30 of Sarma[53] for a similar example in Śivasaṅkalpopaniṣat

42. However S.A. Paramahans[46] cites a counter example from Ṛgveda. The hymn 3.9.9 reads as *trīṇi śatāni trī sahasrāṇy agnim trimśac ca devānava cāsaparyan*, which refers to the number 3339 being expressed as three hundred three thousand and thirty and nine, apparently not in coherence with the place-value notation and more akin to additive principle. This motivates him to conclude, '*...ancient Indians introduced the decimal place-value after the early Vedic period....*'. But in presence of so many examples in tune with decimal place-value system found in various early Vedic texts, only a tiny fraction of which is cited in this article, we would rather take this one as a stray incident, occurred perhaps due to some metrical compulsion of composition.

43. Kaplan[34] argued at length to establish that this is a direct example of Greek influence, referring to the famous problem known as '*sand-reckoner*' by Archimedes (born c.287 BCE). However one must observe that, while the sequence of numbers occurring in the story of Buddha is purely decuple in nature, what Archimedes cooked up as numbers of high order in sand-reckoner was based on the clumsy Greek system of enumeration, in which the largest name of a number was 'myriad', merely 10^4. Of course, it was sheer genius of Archimedes that made him declare a myriad myriad as a number of first order and then taking it as unit and repeating the similar process phase after phase, he proceeded further to higher and higher orders of numbers. But this does show, conceptually how very far away

from a place-value system, let alone decimal, he was working! Kaplan highlighted the apparently proud exaltation of Buddha, uttered in the moment of his ultimate glory "beyond it is the incalculable" to declare, '*there is no notion yet of a full positional system (if there were, we would see that enumeration cannot end........)*'[34] [pp. 40]

44. In the words of Severus Sebokht, a Syrian monk(662 CE) : *I shall not now speak of the knowledge of the Hindus,....of their methods of calculation which no words can praise strongly enough—I mean the system using the nine symbols.*[4]

In the words of a Spanish Monk, Vigilán,(976 CE), found in a writing referred to as the Codex Vigilanus, now kept in the library of the monastery of El Escorial, Spain[15] : *The Indians have an extremely subtle intelligence....The best proof of this is the nine symbols with which they represent each number no matter how large.*

45. One finds from his writings that he gained knowledge from *Kusumapura*, a place believed to be near Patna of Bihar in modern India.

46. *Āryabhaṭīya, Gaṇitapāda*, verse 2

47. Jainism arose as a revolt against the Vedic practice of sacrifice and is perhaps as old as the Vedas, if not older. Once it was thought that a contemporary of *Gautam Buddha*, *Vardhamāna Mahāvīra* by name, was the founder of jainism. But now historians are certain that he was only the last *Tīrthankara*. An interesting aspect of the Jaina philosophical concept of zero as a number and a non-number is found in *Śrī Bhuvalay of Kumudendu*, in a narration of an ancient legend relating to the first *Tīrthaṇkara*, *Ṛṣabhadeva*.[57] The Jainas had a rich tradition of mathematics, mostly known from the later commentaries, many an original text being not extant.[61]

48. For an almost unending array of examples one may refer to Ifrah.[25]

49. Often referred to as the Father of Algebra, he worked in the House of Wisdom (*Bait Al-Hikma*), an institution created in Baghdad for the advancement of philosophy, astronomy and mathematics, during the rule of Al-Ma'mun, the Caliph who is mentioned in the famous *One thousand and one Arabian Nights*. His treatise on Arithmetic, called *De Numero Indorum*, is known to us through a XIII century version in Latin, as its Arabic version has not survived.

50. Around 750 CE in the court of Caliph Al-Mansoor, a copy of an Indian mathematical work, which some historians argue to be *Brāhmasphuṭasiddhānta* by Brahmagupta, was presented and later translated by the name *Sindhind* and presumably this motivated the arithmetical work of Al-Khwarizmi.

51. The first known occurance of the modern term 'zero' is found in an Italian book *De arithmetica opusculum by Philippi Calandri*, pulished in 1491 in Florence.

52. *Piṅgala* employs *bhūtasaṃkhyā* (i.e., word numerals) rather frequently. But in all these places he uses the words singly. Therefore, his employment of word numerals does not clearly indicate place-value.

53. The word comes from the root √pigi, derived as *piṅgayati iti pi ṅgala*, meaning *that which gives light*[56]

54. It is not settled beyond doubt whether this priest is the same person as the prosodist under

discussion. Indeed, going perhaps a bit too far, Van Nooten observed "*we do not know who the author Piṅgala was, we do not know where he lived, when his work was composed and finally, whether the work going by his name was really all his, or a product of his school, or a conglomerate of texts fragments assembled at one time and then forth transmitted under his name.*" However, though the first two doubts are shared by many scholars, the final one is not corroborated in their findings.

55. It is to be noted that *śutras* are mostly prose aphorisms, though one may find some *śutras* in versified form as well.

56. There are two kinds of derivations of the word *chandas*, '*laukika*' (earthly) and '*alaukika*' (unearthly). In the Nirukta of Yāska one finds '*chandāṁsi chādanāt iti*' i.e., 'chandas' comes from the root √*chad*, which means 'to cover'; in the grammar by *Pāṇini*, the word comes from the root √cand, derived as *candayati hlādayati iti chandaḥ cander ādeśaś ca chaḥ*. [56] (pp.51) But what were they suppose to 'cover'? In the *Aitareya Āraṇyaka, adhyāya I, khaṇḍa V, āraṇyaka II*, one may find in the context of worshiping the *prāṇa* God, that He is to be meditated as covered by the meters, whence the meters are *chandas*. It states further, *chādayanti ha vā enaṃ chandāṁsi pāpāt karmaṇo yasyāṁ cid api.kāmayate ya evam etac chandasāṃ chandastvaṃ veda iti*, which means, *he who knows meters as the cover of the prāṇa God will be saved from his sins* [56] (pp.52).

57. The German Indologist Albrecht Weber was the first to make a critical study of the *Chandaḥsūtra* as early as 1863, under the title "*Über die Metrik der Inder*" in the eighth volume of the journal Indische Studien.

58. Piṅgala in his work mentioned the names of Tāṇḍī, Saitava, Kāśyapa, Kātyāyana, Māṇḍavya, Raṭi etc.

59. While dealing with these constructions, *Piṅgala* considered only those meters where all the four feet of the verse have identical pattern, called *samavṛttas*. A detailed lucid account of the actual calculations may be found in [42], [61], [55]

60. The bold letters indicate actual word by word translation, whereas the completion of the sentences are done according to the commentary of *Halāyudha*. This is a good example to cite how terse the presentations of *sūtras* used to be.

61. *rūpa*, meaning 'form' or 'appearance', used to stand for 'one'; this word is synonymous with '*tanu*', the 'body', as a symbol for number 'one'.

62. However, his treatise does not provide us with any representation of numerals. Wherever he had referred to *śūnya*, he did it in language and not by any symbol. So whatever symbol for *śūnya* he might have had in his mind, remains beyond us.

63. The passage where the binary system developed is in all likelihood part of the original work[42] (pp.32)

64. Sarma observes that[55], "*The Pratyaya section of the eighth chapter is not a loose appendage but is anticipated in the preceding chapters of Chandaḥsūtra.*" Indeed, the treatise begins with the definition of the eight *gaṇas* or triplets made of all possible permutations with two kinds of syllables,

laghus taken as *l* and *gurus* taken as *g*. They are then given names as m, the *ma-gaṇa* consisting of the arrangement *ggg*; then comes *y*, the *ya-gaṇa* consisting of the arrangement *lgg*; next is *r*, the *ra-gaṇa* consisting of the arrangement *glg*; etc. Sarma[55] claims that, *"this sequence of triplets make sense only in the context of the Prastāra of the eighth chapter........otherwise, Piṅgala could have enumerated the triplets in any other sequence."*

65. *jyeṣṭhabhrātṛbhir vihite vyākaraṇe anujas tadbhagavān piṅgalācāryas tan matam anubhāvya śikṣāṃ vaktuṃ pratijānīte*

66. One may find an interesting discussion about the connotation of *adarśanaṃ* in this regard with respect to its verbal root *dṛś* in a recent article by Staal[62] (pp. 45).

67. The smallest bit of language that has its own meaning, either a word or a part of a word, vide, Cambridge Advanced Learner's Dictionary.

68. Pandit[44] however gives a much wider perspective to the term '*adarśanaṃ*' than merely 'non-appearance'. Quoting the commentary by *Kāśīkā*, he points out that, the word *adarśana* is to be interpreted as follows:

adarśanam aśravaṇam anuccāraṇam anupalabdhir abhāvo varṇavināśa ity anarthāntaram
etaih śabdair yo'rtho'bhidhīyate tasya lopa itīyam saṁjñā bhavati

When a sound is not heard, pronounced, available or existing or is lost anywhere, even if its place there is due on that occasion, it is said to be lupta i.e., disappeared, or in other words, 'to have amounted to zero'.

69. In the commentary by *Kāśika*, one may find in this regard : *prasaktasyādarśanaṁ lopasaṁjñaṁ bhavati.*

70. In exact language of Pandit[44], "*...as in 125 and 105 in which the zero in the later indicates the place of decimal and can compare with the decimal place number 2 in the former. Though numerically zero is not equal to or identical with the number 2, from the point of view of its capacity as an indicator of the place, it is fully on par with the later. In the same way, though phonetically the, śap = 0, it is on par with non-zero counterpart so far as its place-value is concerned.*"

71. Seven of them are given, of which we present only five, others being fairly technical.

72. One may find the photographs of many relevant papyri in this regard in the article entitled *Consideration of the Greek symbol 'zero'* by Raymond Mercier, available in the internet. http//:/ anubis.dkuug.dk/JTC1/SC2/WG2/docs/.2708.pdf

73. Earlier, this sign, often referred to as the Hellenistic zero, used by astronomers like Ptolemy (c. 140 CE) and some of his successors, only in the fractional part of a sexagesimal number (called minutes, seconds, third, fourth etc.) and not in the integral part, was thought to have emerged out of the first letter 'omicron' of the Greek word οῦδέν ('*ouden*'), meaning nothing. However O. Neugebaur[41] dismisses this claim by pointing out the fact that 'omicron' was already used to mean 70 according to the Greek alphabetic enumeration system and A. Jones[27] opined in his favour. Some other scholars believe that it came from '*obol*', a contemporary coin of almost no value. They conjecture that the

typical round sign evolved during the use of counters in sand-board used for arithmetical calculation. The impression left on sand when such a counter was removed to leave an empty column, perhaps gave birth to the symbol.[34]

74. It however does not explain why the Greek mathematicians did not or could not use this so called 'zero' to the full potential. They were the natural inheritors of Babylonian place-value system, due to their geographical location, yet even after they were exposed to this more powerful system of enumeration, they apparently failed to judge the potential of it, used it only in astronomical calculation, and that too not freely and arithmetically stuck to their own clumsy system of enumeration. Why is it that the genius of Archimedes or the galaxy of geometrical giants could not recognise it? Some believe that it was perhaps due to their over emphasis on geometric ideas, trying to interpret the universe in tandem with their perception of geometry, and this perception did not approve of the concept of 'zero'. Or was it the Aristotelian dictum—'Nature abhors vacuum' that stood in their way, raising a philosophical conundrum like 'How can nothing be represented by something?'

75. "*India's sands were never so kind to her records as Babylonia*'s sands have been to her clay tablets."[2]

76. One may quote G.R. Kaye (1915), as found in pp. 215 of Jacob[28] : '*The achievements of Greeks in mathematics and art form the most wonderful chapters in the history of civilization, and these achievements are the admiration of western scholars. It is therefore natural that western investigators in the history of knowledge should seek for traces of Greek influence in later manifestations of art, and mathematics in particular.*' However, Martin Bernal opines to the contrary. One may refer to his book, *Black Athena. The Afroasiatic Roots of Classical Civilization*, Vol. 1, *The Fabrication of Ancient Greece* 1785-1985, Vintage, London, (1991)

77. While someone claims that '*the mathematical conception of zero....was also present in the spiritual form from 17000 years back in India*'[37], some other seems to be convinced that *Bakshālī* manuscript must belong to 12th century CE !

78. The exact quotation is in the book, *Elements of Mathematics*: Theory of Sets, Springer 1968

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ON G_H-MODULES

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ABSTRACT : In this paper we introduce and study the notion of a class of hypermodulcs (called G_H-module) over G_H-ring. We obtain the equivalent conditions for a G_H-module to be Noetherian. The correspondence theorems of submodules of G_H-module and also the isomorphism theorems of G_H-module are established.

Key words : G_H-module, simple G_H-module, G_H-module-homomorphism.
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1. INTROCUCTION

Krasner's hyperring [4], introduced and studied by Krasner, M. is a hypercompositional structure $(S, +, \cdot)$ where $(S, +)$ is a canonical hypergroup, $(S, \cdot)$ is a semigroup in which the zero element is absorbing and the operation ' $\cdot$ ' is a two-sided distributive one over the hypercomposition $+$. A hypergroup [2] (or multigroup) (G, o) is a semihypergroup (i.e., a hypercompositional structure with single associative hyperoperation) such that for all $a \in G$, $G \mathrm{~o~} a = a \mathrm{~o~} G = G$. A cononical hypergroup [2] (G, o) is a commutative ($a \mathrm{~o~} b = b \mathrm{~o~} a$, $\forall a, b \in G$) hypergroup with a scalar identity e (in the sense that $e \mathrm{~o~} a = \{a\}$, $\forall a \in G$), of which every element has a unique inverse $a' \in G$ (in the sense that $e \in a \mathrm{~o~} a'$) and is also invertible (in the sense that $b \in a \mathrm{~o~} x \Rightarrow x \in a' \mathrm{~o~} b$, $\forall a, b, x \in G$). Chaopraknoi, S. and Kemprasit, Y. introduced in 2005 the notion of semihyperring $(S, +, \cdot)$ [1] where $(S, +)$ is a semihypergroup, $(S, \cdot)$ is a semigroup and the operation ' $\cdot$ ' is both left and right distributive accross the hyperopreration $+$. Interchanging the modes of the operations involved in the hyperstructure semihyperring, we defined in [9] another class of hyperstructure called hypersemiring which is an (additive) commutative semigroup $(S, +)$ endowed with a hyperoperation $\mathrm{o} : S \times S \to P^*(S)$ $(= P(S) \setminus \{\phi\})$ such that for all $x, y, z \in S$, (i) $x \mathrm{~o~} (y \mathrm{~o~} z) = (x \mathrm{~o~} y) \mathrm{~o~} z$ and (ii) $(x + y) \mathrm{~o~} z = x \mathrm{~o~} z + y \mathrm{~o~} z$, $x \mathrm{~o~} (y + z) = x \mathrm{~o~} y + x \mathrm{~o~} z$ (where for any $A, B \in P(S)$, $A + B = \{a + b : a \in A, b \in B\}$). In contrary to Krasner's hyperring, another kind of hyper-ring was introduced in [9]. This hyper-ring is a hypersemiring $(S, +, \mathrm{o})$ where $(S, +)$ is a commutative group whose identity element 0_S is absorbing in the hypersemiring $(S, +, \mathrm{o})$ (in the sense that $0_S \in 0_S \mathrm{~o~} x = x \mathrm{~o~}$

0_S, $\forall x \in S$). G_H-ring $(R, +, \text{o})$[8] is a generalised hyper-ring [9] which being assumed to have absorbing zero 0_R (in the sense, $0_R \text{ o } x = x \text{ o } 0_R = \{0_R\}$, instead of $0_R \in 0_R \text{ o } x = x \text{ o } 0_R$) is defined by replacing 'set equality' type distributive axioms of hyper-ring by 'one-sided containment ($\subseteq$)' type distributive axioms $(x + y) \text{ o } z \subseteq x \text{ o } z + y \text{ o } z$ and $x \text{ o } (y + z) \subseteq x \text{ o } y + x \text{ o } z$, $\forall x, y, z \in R$. A non-empty finite subset $\mathcal{E}_l$(resp. $\mathcal{E}_r$) $= \{e_1, e_2, \ldots, e_n\}$ of a G_H-ring $(R, +, \text{o})$ is called a left (resp. right) identity set (in short i-set) [8] of R if (i) $e_i \neq 0_R$ for atleast one $i = 1, 2, \ldots, n$, and (ii) for any $a \in R$, $a \in \sum_{i=1}^{n} e_i \text{ o } a$ (resp. $a \in \sum_{i=1}^{n} a \text{ o } e_i$). A non-empty finite subset $\mathcal{E}$ of a G_H-ring $(R, +, \text{o})$ is an i-set of R if it is both a left i-set and a right i-set of R. an element e of a G_H-ring $(R, +, \text{o})$ is called a (left, right) hyperidentity of R if the set $\{e\}$ is an (resp. left, right) i-set of R.

This is proved in [8] that in case of a G_H-ring $(R, +, \text{o})$, $(-x) \text{ o } y \cap (-(x \text{ o } y)) \neq \phi$ and $x \text{ o } (-y) \cap (-(x \text{ o } y)) \neq \phi$ for any x, y in R, where for any $A \in P^*(R)$, $-A = \{-a : a \in A\}$. Unlike a ring, the equality of the set-expressions $(-x) \text{ o } y$, $x \text{ o } (-y)$ and $-(x \text{ o } y)$ does not hold in general, on a G_H-ring $(R, +, \text{o})$ for any $x, y \in R$. It is thus considered and studied in [8] a particular class of G_H-ring, called the G_H-rings with condition $(\mathcal{R})$ (i.e., $(-x) \text{ o } y = x \text{ o } (-y) = -(x \text{ o } y)$ $\forall x, y \in R$).

In 1982, R. Rota initiates the study of multiplicative hyperring [6] which is an additive commutative group $(R, +)$ endowed with a hyperoperation o which satisfies the first two axioms of G_H-ring alongwith the condition $(\mathcal{R})$. It is thus clear from the definitions that the class of G_H-rings with condition $(\mathcal{R})$ is precisely that of multiplicative hyperrings with absorbing zero.

Fotea, V. L. considers in [3] a hyperring $(R, \uplus, \text{o})$, where $(R, \uplus)$ is a commutative hypergroup, (R, o) is a semihypergroup and the hyperoperation o is distributive over $\uplus$. Over this type of hyperring $(R, \uplus, \text{o})$, Fotea defines a commutative hypergroup $(M, \oplus)$ to be a hypermodule if there exists a mapping $\odot : R \times M \rightarrow P^*$ (described by $(r \odot a) \rightarrow r \odot a$) satisfying for any $r, s \in R$ and $a, b \in M$, (i) $(r \text{ o } s) \odot a = r \odot (s \odot a)$, (ii) $r \odot (a \oplus b) = r \odot a \oplus r \odot b$, (iii) $(r \uplus s) \odot a = r \odot a \oplus s \odot a$. Fotea studies the connection between the hypermodules and fuzzy hypermodules and also extends the works fo Sen, Ameris and Chowdhury on Fuzzy Hypersemigroups [7].

Massouros, Ch. G. studies the free and the cyclic hypermodules [5] over Krasner's hyperrings in 1988. Massouros' hypermodule over a krasner's hyperring $(R, +, \cdot)$ is a canonical (additive) hypergroup $(M, +)$ together with, a mapping $\text{o} : R \times M \rightarrow M$ (described by (r, a)

$\to r \circ a$) such that for any $r, s \in R$ and $a, b \in M$, (i) $(r \cdot s) \circ a = r \circ (s \circ a)$, (ii) $r \circ (a+b) = r \circ a + r \circ b$, (iii) $(r + s) \circ a = r \circ a + s \circ a$, (iv) $r \circ 0 = 0$. If, in addition, the hyperring R has 1, then hypermodule M is said to be unitary if for any $a \in M$, $1 \circ a = a$.

In contrary to Massouros' hypermodule, we have considered here another type of hypermodule (called G_H-module) over a G_H-ring, which is simply an additive commutative group M together with a mapping $\circ : R \times M \to P^*(M)$ satisfying some axioms as stated formally in te definitioon 2.1.

2. PROPERTIES OF G_H-MODULES

Let $R = M_{2\times3}(\mathrm{R})$ be the set of all 2×3 real matrices and $A = \{\alpha, \beta\} \subseteq M_{3\times2}(\mathrm{R})$. Then, $(R, +, \circ)$ is a G_H-ring where + is the usual addition of matrices and o is the hyperoperation on M defined by $a \circ b = \{a{\cdot}\alpha{\cdot}b,\ a{\cdot}\beta{\cdot}b\}$, in which ' $\cdot$ ' is the usual matrix multiplication. Now let $M = M_{2\times2}(\mathrm{R})$ be the set of all 2×2 real matrices. Then, $(M, +)$ is a commutative group where + is the usual addition of matrices. We consider the mapping $\circ : R \times M \to P^*(M)$, described by $(r, a) \to r \circ a$, where $r \circ a = \{r{\cdot}\alpha{\cdot}a,\ r{\cdot}\beta{\cdot}a\}$. Clearly here $(r \circ s) \circ a = r \circ (s \circ a)$, $\forall r, s \in R$ and $\forall a \in M$. Also, this is to be observed that, $p \in r \circ (a + b) \Rightarrow p = r{\cdot}\alpha{\cdot}(a + b)$ (for some $\alpha \in A$) $= r{\cdot}\alpha{\cdot}a + r{\cdot}\alpha{\cdot}b \in r \circ a + r \circ b$, $\forall r \in R$ and $\forall a, b \in M$. Thus, $r \circ (a + b) \subseteq r \circ a + r \circ b$. Similarly, $(r + s) \circ a \subseteq r \circ a + s \circ a$, $\forall r, s \in R$ and $\forall a \in M$. Moreover, corresponding to $A = \left\{\begin{bmatrix}1&0\\0&0\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\\0&0\end{bmatrix}\right\}$, we see that, for $r = \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix} \in R$, $a = b = \begin{bmatrix}0&1\\0&0\end{bmatrix} \in M$, $r \circ (a + b) = \left\{\begin{bmatrix}0&2\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$; while, $r \circ a = r \circ b = \left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$ and so, $r \circ a + r \circ b = \left\{\begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&2\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$, whence, $r \circ (a + b) \subsetneq r \circ a + r \circ b$. Again for $r = s = \begin{bmatrix}1&0&0\\0&0&0\end{bmatrix} \in R$ and for $a = \begin{bmatrix}1&0\\0&0\end{bmatrix} \in M$, $(r + s) \circ a = \left\{\begin{bmatrix}2&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$; while, $r \circ a = s \circ a = \left\{\begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}\right\}$ and so, $r \circ a + s \circ a =$

$\left\{\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}2 & 0\\0 & 0\end{bmatrix}, \begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}\right\}$, whence, $(r+s) \text{ o } a \subsetneq r \text{ o } a + s \text{ o } a$. This evokes the idea to perceive formally and study generally a new class of hyperstructure (called the G_H-module over G_H-ring) defined as follows:

Definition 2.1. Let $(R, +, \text{o})$ be a G_H-ring. A (left) G_H-module over G_H-module over G_H-ring R(R-G_H-module, in short) is a commutative group $(M, +)$ together with a mapping $\text{o} : R \times M \to P^*(M)$, called an external hypercomposition (described by $(r, a) \to r \text{ o } a$) such that for any $r, s \in R$ and $a, b \in M$,

(i) $(r \text{ o } s) \text{ o } a = r \text{ o } (s \text{ o } a)$, (ii) $r \text{ o } (a + b) \subseteq r \text{ o } a + r \text{ o } b$,

(iii) $(r + s) \text{ o } a \subseteq r \text{ o } a + s \text{ o } a$, (iv) $r \text{ o } 0_M = 0_R \text{ o } a = \{0_M\}$.

If in addition the G_H-ring R has an i-set $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ (resp. a hyperidentity e) then the G_H-module $(M, +, \text{o})$ over a G_H-ring $(R, +, \text{o})$ is said to be unitary (resp. hyperunitary) with respect to $\mathcal{E}$ (resp. e) if for any $a \in M$, $a \in \sum_{i=1}^{n} e_i \text{ o } a$ (resp. $a \in e \text{ o } a$).

Example 2.2. (a) Every G_H-ring $(R, +, \text{o})$ (with an i-set $\mathcal{E}$ or a hyperidentity e) is a (resp. umitary or hyperunitary) G_H-module (resp. with respect to $\mathcal{E}$ or with respect to e) over itself, with respect to the external hypercomposition $\odot$ defined by $r \odot a = r \text{ o } a$, $\forall r, a \in R$.

(b) Corresponding to every subset $A \in P^*(R)$ (with $|A| \geq 2$) of a ring $(R, +, \cdot)$, there exists a G_H-ring $(R_A, +, \text{o})$ where $R_A = R$ and for any $x, y \in R_A$, $x \text{ o } y = \{x{\cdot}a{\cdot}y : a \in A\}$. This G_H-ring is called the G_H-ring over the ring R induced by A, which we have studied in [8]. Now let M be an R-module. Then the additive group $(M, +)$ is a G_H-module over the G_H-ring R_A where the external hypercomposition o is defined by $r \text{ o } a = \{r{\cdot}\alpha{\cdot}a : \alpha \in A\}$. If the ring R has a unity and M is a unitary R-module and if R_A has a hyperidentity e, then M is a hyperunitary G_H-module over G_H-ring R_A.

(c) Let $R = \{ a\sqrt{2} + b\sqrt{3} : a, b \in \mathbb{Q}\}$ and $A = \{\sqrt{2}, \sqrt{3}\}$. Then, with respect to usual addition + of reals, $(R, +)$ is a commutative group, with identity 0. On R, o is a hyperoperation defined by

$$\left(a\sqrt{2} + b\sqrt{3}\right) o \left(c\sqrt{2} + d\sqrt{3}\right) = \left\{\left(a\sqrt{2} + b\sqrt{3}\right).t.\left(c\sqrt{2} + d\sqrt{3}\right) : t \in A\right\}.$$

$$= \left\{p\sqrt{2} + 2q\sqrt{3}, 3q\sqrt{2} + p\sqrt{3}\right\},$$ where $p = 2ac + 3bd$ and $q = bc + ad$.

Then, $(R, +, \circ)$ is a commutative G_H-ring, in which $\varepsilon = \left\{\sqrt{2}+\sqrt{3}, -\sqrt{2}-\sqrt{3}\right\}$ is an *i*-set. The additive group of reals R is a unitary G_H-module over the aforesaid G_H-ring R where for any $r \in R$ and $a \in \mathrm{R}$, $r \circ a = \{r\cdot\sqrt{2}\cdot a, r\cdot\sqrt{3}\cdot a\}$.

Definition 2.3. Let M and N be two G_H-modules over a G_H-ring R. A mapping $f : M \to N$ is called an R-G_H-module homomorphism if for any $r \in R$ and $a \in M$, (i) $f(a + b) = f(a) + f(b)$, (ii) $f(r \circ a) \subseteq r \circ f(a)$, where for any $A \in P^*(M)$, $f(A) = \{f(a) : a \in A\}$.

Example 2.4. Let R be a ring and M and N be two R-modules. Then, for any $A \in P^*(R)$ (with $|A| \geq 2$), as we have seen in the example 2.2(b), the aditive groups M and N are two G_H-modules over G_H-ring R_A. This is to be observe that every R-module homomorphism from M to N is an R_A-G_H-module homomorphism. The converse of this assertion is not true. In fact, let R be a non-commutative division ring with $|R| \geq 3$ and M be a non-trivial vector space (i.e., $R\ M \neq \{0\}$) over R. Then the mapping $f : M \to M$ defined by $f(a) = x \cdot a$, for some $0 \neq x \in R$ is a group isomomorphism from the additive group M onto itself. Since R is non-commutative, f is not an R-morphism. Now let $A = R\setminus\{0\}$. Then, f is a R_A-G_H-module homomorphism. Indeed, for any $r \in R$ and $a \in M$, $p \in f(r \circ a) = f\{r \cdot s \cdot a : s \in A\} \Rightarrow p = x \cdot r \cdot s \cdot a = r.(r^{-1}x \cdot r \cdot s \cdot x^{-1}) \cdot x \cdot a \in r \cdot t \cdot x \cdot a \in r \circ (x \cdot a)$ (since, $t = r^{-1} \cdot x \cdot r \cdot s \cdot x^{-1} \in A) \subseteq r \circ f(a)$, whence, $f(r \circ a) \subseteq r \circ f(a)$.

Definition 2.5. Let $(M, +, \circ)$ be a G_H-module over a G_H-ring R. A non-empty subset N of M is called a submodule of the G_H-module M if

(i) N is an additive subgroup of the group M,

(ii) for any $r \in R$ and $b \in N$, $r \circ b \subseteq N$.

Proposition 2.6. If N and K be two submodules of a G_H-module M, then $N \cap K$ and $N + K$ are also submodules of the G_H-module M. Moreover $N \cap K$ is a submodule of both the G_H-modules N and K, each of which again being submodules of the G_H-module $N + K$.

Proof. We shall only consider the case of $N + K$ for two submodules M and N. Clearly here $N + K$ is a subgroup of the additive group M. Now, for any $r \in R$ and for any $n \in N$, $k \in K$, $r \circ n \subseteq N$ and $r \circ k \subseteq K$, whence $r \circ (n + k) \subseteq r \circ n + r \circ k \subseteq N + K$.

Proposition 2.7. Let M and N be two G_H-modules over G_H-ring R and $f : M \to N$ be an R-G_H-module homomorphism. Then, for any submodule N' of the G_H-module M, $f^{-1}(N') = \{a \in M : f(a) \in N'\}$ is a submodule of the G_H-module M.

Proof. Since, N' is a submodule of the G_H-module N, so N' is a subgroup of the additive group M. Thus, for the R-G_H-module homomorphism f, $f^{-1}(N')$ is a subgroup of the additive group M (since, f is a group homomorphism). Again $r \in R$ and $a \in f^{-1}(N') \Rightarrow f(a) \in N' \Rightarrow f(r \text{ o } a) \subseteq r \text{ o } f(a) \subseteq N' \Rightarrow r \text{ o } a \subseteq f^{-1}(N')$. Thus, $f^{-1}(N')$ is a submodule.

Corollary 2.8. Let $(M, +, \text{o})$ and $(N, +, \otimes)$ be two G_H-modules over a G_H-ring $(R, +, \text{o})$ and $f : M \to N$ be an R-G_H-module homomorphism. Then, the kernel of f (i.e., the set Ker. $(f) = \{a \in M : f(a) = 0_N\}$) is a submodule of the G_H-module M.

Proof. Since, $\{0_N\}$ is a submodule of the G_H-module N, so it follows from proposition 2.7 that Ker.$(f) = f^{-1}(\{0_N\})$ is a submodule of M.

Definition 2.9. A G_H-module $(M, +, \text{o})$ over a G_H-ring $(R, +, \text{o})$ is said to satisfy the condition $(\mathcal{M})$ if the set equality $(-r) \text{ o } a = r \text{ o } (-a) = -(r \text{ o } a)$ (called the condition $(\mathcal{M})$) holds true for any two elements $r \in R$ and $a \in M$.

Example 2.10. For any commutative group $(G, +)$, the hyperstructure $(G, +, \text{o})$ is a commutative G_H-ring when $x \text{ o } y = \{0_G, x, y\}$, $\forall x, y \in G \setminus \{0_G\}$, and $x \text{ o } 0_G = 0_G \text{ o } x = \{0_G\}$, $\forall x \in G$. The G_H-ring G over itself is a G_H-module which does not satisfy the condition $(\mathcal{M})$.

Now we define $\odot : G \times G \to P^*(G)$ by stating that $g \text{ o } a = \{0_G, g, -g, a, -a\}$, $\forall g, a \in G \backslash \{0_G\}$, and $g \odot 0_G = 0_G \odot a = \{0_G\}$, $\forall g, a \in G$. Then the additive group G is a G_H-module with condition $(\mathcal{M})$, over the G_H-ring $(G, +, \text{o})$ with respect to the external hypercomposition $\odot$.

Definition 2.11. Let A be a subset of a G_H-module $(M, +, \text{o})$ over a G_H-ring $(R, +, \text{o})$. Then, the submodule of the G_H-module M generated by A is the smallest submodule of the G_H-module M containing A which is denoted by $<A>$. A submodule M' of the G_H-module $(M, +, \text{o})$ is said to be finitely generated if $M' = <A>$ for some finite subset A of M. The cyclic submodule of the G_H-module M generated by an element a of M, denoted by (a) is the submodule $<\{a\}>$ of G_H-module M.

Remark 2.12. The submodule of a G_H-module $(M, +, \text{o})$ over a G_H-ring $(R, +, \text{o})$, generted by the empty subset ϕ of M is the zero module $\{0_M\}$ and thus is the smallest submodule of M.

Proposition 2.13. Let A be a subset of a G_H-module $(M, +, \text{o})$ with condition $(\mathcal{M})$, over a G_H-ring $(R, +, \text{o})$ and $a \in M$. Let I be a left ideal of the G_H-ring $(R, +, \text{o})$. Then, the sets (i) $\overline{I \circ A}$, (ii) $\overline{I \circ a}$, (iii) $\overline{R \circ A}$, (iv) $\overline{R \circ a}$ are submodules of the G_H-module M, where, for any $(S, A) \in P^*(R) \times P^*(M)$ and for any $a \in M$,

$$\overline{S \circ A} = \cup\left\{\sum_{i=1}^{n} s_i \circ a_i : s_i \in S, a_i \in A, n \in \mathrm{N}\right\} \text{ and } \overline{S \circ a} = \overline{S \circ \{a\}}.$$

Proof. Let p, $q \in \overline{I \circ A}$ and $r \in R$. Then, $p \in \sum_{i=1}^{n} r_i \circ a_i$ and $q \in \sum_{j=1}^{m} s_j \circ b_j$ for some r_i, $s_j \in I$ and a_i, $b_j \in A$. So, $p - q \in \left(\sum_{i=1}^{n} r_i \circ a_i\right) - \left(\sum_{j=1}^{m} s_j \circ b_j\right) = \sum_{i=1}^{n} r_i \circ a_i + \sum_{j=1}^{m} -(s_j \circ b_j)$ $\sum_{i=1}^{n} r_i \circ a_i + \sum_{j=1}^{m} (-s_j) \circ b_j \subseteq \overline{I \circ A}$ and also $q \in r \circ p \subseteq r \circ \sum_{i=1}^{n} r_i \circ a_i \subseteq \sum_{i=1}^{n} r \circ (r_i \circ a_i) = \sum_{i=1}^{n} (r \circ r_i) \circ a_i \Rightarrow q \in \sum_{i=1}^{n} t_i \circ a_i \subseteq \overline{I \circ A}$ for some $t_i \in (r \circ r_i) \subseteq I$. Hence $\overline{I \circ A}$ is a submodule of M. The sets (ii), (iii) and (iv) can be proved to be submodules of M similarly.

Proposition 2.14. Let A be a subset of a G_H-module (M, +, o) with condition ($\mathcal{M}$), over a G_H-ring (R, +, o) and $a \in M$. Then,

(i) $\langle A \rangle = \overline{R \circ A} + \cup\left\{\sum_{i=1}^{n} \lambda_i a_i : \lambda_i \in \mathrm{Z}, a_i \in A\right\}$

(ii) $\langle a \rangle = \overline{R \circ a} + \cup\{\lambda a : \lambda \in \mathrm{Z}\}$

If, in addition, the G_H-ring (R, +, o) has an i-set $\mathcal{E}$ and G_H-module (M, +, o) is unitary with respect to $\mathcal{E}$ then.

(i)′ $\langle A \rangle = \overline{R \circ A}$ and (ii)′ $\langle a \rangle = \overline{R \circ a}$

Proof. Let a, $b \in \overline{R \circ A} + \cup\left\{\sum_{i=1}^{m} \lambda_i a_i : \lambda_i \in \mathrm{Z}, a_i \in A\right\}$ and $r \in R$. Then, $a \in \sum_{i=1}^{n} r_i \circ a_i + \sum_{j=1}^{m} \eta_j c_j$ and $b \in \sum_{k=1}^{p} s_k \circ b_k + \sum_{\iota=1}^{q} \iota_\iota d_\iota$ for some r_i, $s_k \in R$, a_i, b_k, c_j, $d_\iota \in A$, η_j, $\iota_\iota \in \mathrm{Z}$ and m, n, p, $q \in \mathrm{N}$. So, $a - b \in \left(\sum_{i=1}^{n} r_i \circ a_i\right) - \left(\sum_{k=1}^{p} s_k \circ b_k\right) + \sum_{j=1}^{m} \eta_j c_j - \sum_{\iota=1}^{q} \iota_\iota d_\iota = \sum_{i=1}^{n} r_i \circ a_i$

$+ \sum_{k=1}^{p} -(s_k \circ b_k) + \sum_{j=1}^{m} \eta_j c_j + \sum_{\iota=1}^{q} (-\iota_\iota) d_\iota = \sum_{l=1}^{n} r_l \circ a_l + \sum_{k=1}^{p} (-s_k) \circ b_k + \sum_{j=1}^{m} \eta_j c_j + \sum_{\iota=1}^{q} (-\iota_\iota) d_\iota$ $\subseteq \overline{R \circ A} + \cup \left\{ \sum_{l=1}^{n} \lambda_l a_l : \lambda_i \in Z, b_l \in A \right\}$ and also $c \in r \text{ o } a \subseteq r \text{ o } \left(\sum_{l=1}^{n} r_l \circ a_l + \sum_{j=1}^{m} \eta_j c_j \right) \subseteq$ $\sum_{i=1}^{n} r \circ (r_l \circ a_l) + \sum_{j=1}^{m} \eta_j (r \circ c_j) = \sum_{i=1}^{n} (r \circ r_l) \circ a_i + \sum_{j=1}^{m} \eta_j (r \circ c_j) \Rightarrow c \in$ $\left(\sum_{l=1}^{n} t_l \circ a_l + \sum_{j=1}^{m} \eta_j (r \circ c_j) \right) + \{0_M\}$ (for some $t_l \in (r \text{ o } r_l)$) $\subseteq \overline{R \circ A} + \cup\{ \sum_{i=1}^{n} \lambda_i a_i : \lambda_l \in Z, a_l \in A\}$. Hence $\overline{R \circ A} + \cup\{ \sum_{l=1}^{n} \lambda_l a_i : \lambda_i \in Z, a_l \in A\}$ is a submodule of M. Clearly, $A \subseteq \overline{R \circ A}$ $+ \cup\{ \sum_{i=1}^{n} \lambda_i a_l : \lambda_l \in Z, a_l \in A\}$. Moreover, for any submodule K with $A \subseteq K$, clearly $\overline{R \circ A}$ $+ \cup\{ \sum_{l=1}^{n} \lambda_i a_l : \lambda_i \in Z, a_i \in A\} \subseteq K$. So, $\langle A \rangle = \overline{R \circ A} + \cup \{ \sum_{l=1}^{n} \lambda_l a_i : \lambda_l \in Z, a_l \in A\}$. The case for (ii) can be proved similarly.

Now, if $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$ be an i-set of the G_H-ring $(R, +, \text{o})$, then for any $a \in A$, $a \in \sum_{i=1}^{n} e_l \circ a \Rightarrow \cup\{ \sum_{l=1}^{n} \lambda_i a_l : \lambda_l \in Z, a_l \in A\} \subseteq \overline{R \circ A}$, whence $\langle A \rangle = \overline{R \circ A}$. The case for (ii)′ can be proved similarly.

Deinition 2.15. An R-G_H-module M is said to be simple if the only submodules of M are $\{0_M\}$ and M.

Proposition 2.16. Let R be a G_H-ring with an i-set and M be a unitary R-G_H-module with condition $(\mathcal{M})$. Then, M is a simple G_H-module if and only if M is a cyclic G_H-module generated by every non-zero element of M.

Proof. Since R is a G_H-ring with an i-set and M is a unitry G_H-module with condition $(\mathcal{M})$, so by proposition 2.14(ii)′, $\langle a \rangle = \overline{R \circ a}$ is a non-zero cyclic submodule of the G_H-module M, for every non-zero element of M. Thus the result follows from the definition 2.15.

Definition 2.17. Let p be an equivalence relation on a G_H-module M over a G_H-ring $(R, +,$

o) and let $A, B \in P^*(M)$. Then we define that, $A\rho B$ if for each $a \in A$ there exists $b' \in$ B such that, $a\rho b'$ holds and for each $b \in B$ there exists $a' \in A$ such that, $a'\rho b$ holds. An equivalence relation ρ on a G_H-module M over a G_H-ring $(R, +, \mathrm{o})$ is said to be a congruence on G_H-module M if for any $r \in R$ and $a, b, c \in M$, $a\rho b \Rightarrow$ (i) $(a + c)\rho(b + c)$ and (ii) $(r \mathrm{~o~} a)\rho(r \mathrm{~o~} b)$.

Proposition 2.18. Let ρ be a congruence relation on a G_H-module M over a G_H-ring R. Let M/ρ be the set of all ρ-eqivalence classes $\bar{a}$ of M under ρ. On M/ρ a binary relation + is defined by $\bar{a}+\bar{b}=\overline{a+b}$, $\forall a, b \in M$. Then $(M/\rho, +)$ is a commutative group. An external hypercomposition o : $R \times M/\rho \to P^*(M/\rho)$ is defined by $r\circ\bar{a} = \overline{r\circ a} = \{\bar{c} : c \in r \mathrm{~o~} a\}$. Then the group M/ρ is a G_H-module over G_H-ring R with respect to the external hypercomposition o

Proof. Since, by definition, ρ is a congruence on the commutative group $(M, +)$, clearly here $(M/\rho, +)$ is a commutative group with identity $0_{M/\rho} = \bar{0}_M$.

Now let $a, b \equiv M$ be such that $\bar{a} = \bar{b}$. Then $a\rho b$. Hence, for any $r \in R$, $(r \mathrm{~o~} a)\rho(r\mathrm{o}b)$. So, for each $c \in r \mathrm{~o~} a$, there is $d' \in r \mathrm{~o~} b$ such that $c\rho d'$ i.e., $\bar{c} = \overline{d'}$. Thus, $r \mathrm{~o~} \bar{a} \subseteq r \mathrm{~o~} \bar{b}$. Similarly, $r \mathrm{~o~} \bar{b} \subseteq r \mathrm{~o~} \bar{a}$, whence $r \mathrm{~o~} \bar{a} = r \mathrm{~o~} \bar{b}$.

For any $r, s \equiv R$ and $a \in M$, $p \in (r \mathrm{~o~} s) \mathrm{~o~} \bar{a} \Rightarrow p \in t \mathrm{~o~} \bar{a}$ (for some $t \in r \mathrm{~o~} s$) $= \overline{t\circ a} \subseteq \overline{(r\circ s)\circ a} = \overline{r\circ(s\circ a)} \Rightarrow p \in \overline{r\circ b}$ (for some $b \in s \mathrm{~o~} a$) $= r \mathrm{~o~} \bar{b} \subseteq r \mathrm{~o~} \overline{s\circ a} = r \mathrm{~o~} (s \mathrm{~o~} \bar{a}) \Rightarrow (r \mathrm{~o~} s) \mathrm{~o~} \bar{a} \subseteq r \mathrm{~o~} (s \mathrm{~o~} \bar{a})$. Similarly, $r \mathrm{~o~} (s \mathrm{~o~} \bar{a}) \subseteq (r \mathrm{~o~} s) \mathrm{~o~} \bar{a}$, whence $(r \mathrm{~o~} s) \mathrm{~o~} \bar{a} = r \mathrm{~o~} (s \mathrm{~o~} \bar{a})$.

Again, $(r + s) \mathrm{~o~} \bar{a} = \overline{(r+s)\circ a} \subseteq \overline{r\circ a+s\circ a} = \overline{r\circ a} + \overline{s\circ a} = r \mathrm{~o~} \bar{a} + s \mathrm{~o~} \bar{a}$. Moreover, for any $a, b \in M$ and $r \in R$, $r \mathrm{~o~} \left(\bar{a}+\bar{b}\right) = r \mathrm{~o~} \overline{a+b} = \overline{r\circ(a+b)} \subseteq \overline{r\circ a+r\circ b} = \overline{r\circ a} + \overline{r\circ b} = r \mathrm{~o~} \bar{a} + r \mathrm{~o~} \bar{b}$.

Finally, for any $a \in M$, $0_R \mathrm{~o~} \bar{a} = \overline{0_R\circ a} = \overline{\{0_M\}}$ (since, $0_R \mathrm{~o~} a = \{0_M\}$) $= \{\bar{0}_M\} = \{0_{M/\rho}\}$. Also, for any $r \in R$, $r \mathrm{~o~} 0_{M/N} = r \mathrm{~o~} \bar{0}_M = \overline{r\circ 0_M} = \overline{\{0_M\}} = \{\bar{0}_M\} = \{0_{M/\rho}\}$.

Definition 2.19. Let N be a submodule of a G_H-module M over a G_H-ring R. For $a, b \in M$, a is said to be congruent to b modulo N, denoted by $a \equiv b(mod\ N)$ if $a - b \in N$.

Definition 2.20. Let N be a submodule of a G_H-module M over a G_H-ring M. Then, $\equiv (mod\ N)$ is a congruence relation on the G_H-module M and $M/_{\equiv(modN)} = \{a + N : a \in M\}$.

Proof. Since N is a submodule of the G_H-module M, so $(N, +)$ is a subgroup of the commutative group $(M, +)$. Hence the relation $\equiv (modN)$ is a congruence relation on the group $(M, +)$ such that $M/_{\equiv(modN)} = \{a + N : a \in M\}$. Now, let $a, b \in M$ be such that $a \equiv b(modN)$. Then, $a = b + n$ for some $n \in N$. Thus, for any $r \in R$, $r \circ a = r \circ (b + n) \subseteq r \circ b + r \circ n \subseteq r \circ b + N \Rightarrow$ for any $p \in (r \circ a)$, $\exists q' \in r \circ b$ and $n' \in N$ such that $p = q' + n'$ i.e., $p \equiv q'(modN)$. Similarly, for any $q \in (r \circ b)$, $\exists p' \in r \circ a$ such that, $p' \equiv q(modN)$. Hence, $(r \circ a) \equiv (r \circ b)(mod\ N)$. So, $\equiv (mod\ N)$ is a *congruence on the* G_H-module M over G_H-ring R.

Proposition 2.21. Let ρ be an equivalence relation on a G_H-module M over a G_H-ring R and let $N = ker.(\rho) = \{a \in M : a\rho 0_M\}$. Then, ρ is a congruence relation on the G_H-module M if and only if N is a submodule of M and ρ is $\equiv (modN)$.

Proof. Let ρ be a congruence relation on a G_H-module M. Then, clearly $N = ker.(\rho)$ is a subgroup of the additive group M and ρ is a congruence modulo the subgroup N. Now let $r \in R$ and $a \in ker.(\rho)$. Then, $a\rho 0_M$ and thus $(r \circ a)\rho(r \circ 0_M) = \{0_M\} \Rightarrow p\ \rho\ 0_M$, $\forall p \in r \circ a \Rightarrow r \circ a \subseteq ker.(\rho)$, whence $N = ker.(\rho)$ is a submodule of the G_H-module M.

The converse part of the proposition has been proved in proposition 2.20.

Remark 2.22. In view of the proposition 2.21, we can say that the class of all congruence relations on a G_H-module M is in one-to-one correspondence with the class of all submodules of M. Thus, corresponding to every submodule N of a G_H-module M over a G_H-ring R, we have the quotient G_H-module $M/N = M/_{\equiv(modN)} = \{a + N : a \in M\}$, where for any, $a, b \in M$ and $r \in R$, $(a + N) + (b + N) = (a + b) + N$ and $r \circ (a + N) = (r \circ a) + N = \{c + N : c \in r \circ a\}$.

Theorem 2.23. (Correspondence theorem)

Let N be a submodule of an R-G_H-module M. Then, there is an inclusion preserving bijection from the set of submodules of M/N to the set of submodules A of M such that $N \subseteq A$.

Proof. N being a submodule of the R-G_H-module M, is a (normal) subgroup of the additive (commutative) group M. Thus the mapping f defined by $A \to f(A) = A/N$ is an inclusion presereving bijection from the set of subgroups A of the additive commutative group M, containing N to the set of subgroups of the group M/N.

Now, let S_M be the set of all submodules A of the G_H-module M such that $N \subseteq A$. while $S_{M/N}$ be the set of all submodules of the quotient G_H-module M/N. Then for any $A \in S_M$, A/N is a subgroup of the additive group M/N (since, A is a sub-group of he group M). Now, for any $r \in R$, and $a + N \in A/N$, we see that $r \circ (a + N) = r \circ a + N \subseteq A/N$ (Since $r \circ$

$a \subseteq A$). So, $A/N \in S_{M/N}$. Again, for any $P \in S_{M/N}$, $X = \{x \in M : x + N \in P\}$ is clearly a subgroup of the group M. Moreover, for any $r \in R$ and $x \in X$, $a \in r \circ x \Rightarrow a + N \in r \circ x + N = r \circ (x + N) \subseteq P$ (since P is a submodule of M/N) $\Rightarrow a \in X$. So, $r \circ x \subseteq X$, $\forall x \in X$. Clearly $N \subseteq X$ (Since, $0_{M/N} = N \in P$). Thus, $X \in S_M$ such that $X \to f(X) = P$. Hence, f is an inclution preserving bijection from S_M to $S_{M/N}$.

Theorem 2.24. (Isomorphism theorems on G_H-modules)

(I) *Let $f : M \to N$ be an epimorphism from a G_H-module $(M, +, \circ)$ to a G_H-module $(N, +, \circ)$ over a G_H-ring $(R, +, \circ)$. Then,*

$$M \,/\, Ker.(f) \simeq N.$$

(II) *Let A and B be two submodules of a G_H-module $(M, +, \circ)$ over a G_H-ring $(R, +, \circ)$.* Then

$$A \,/\, (A \cap B) \simeq (A + B)/B.$$

(III) *Let A and B be two submodules of a G_H-module $(M, +, \circ)$ over a G_H-ring $(R, +, \circ)$, such that $B \subseteq A$. Then*

$$(M/B) \,/\, (A/B) \simeq M/A.$$

Proof. (I) Since, $f : (M, +, \circ) \to (N, +, \circ)$ be an R-G_H-module-epimorphism, so $f : (M, +) \to (N, +)$ is a group-epimorphism. Thus, the mapping $\Psi : (M/\text{ker.}(f), +) \to (N, +)$ defined by $\Psi(a + \text{ker.}(f)) = f(a)$ is a group-isomorphism. Now, for any $a \in M$ and $r \in R$, $\Psi(r \circ (a + \text{ker.}(f))) = \Psi((r \circ a) + \text{ker.}(f)) = f(r \circ a) \subseteq r \circ f(a) = r \circ \Psi(a + \text{ker.}(f))$. Thus, Ψ is an R-G_H-module-isomorphism.

(II) Since, the mapping $f : A \to (A + B)/B$ defined by $f(a) = a + B$, $\forall a \in A$ is a G_H-module-epimorphism with $\text{ker.}(f) = A \cap B$, the proof follows from (I).

(II) Since, the mapping $f : M/B \to M/A$ defined by $f(m + B) = m + A$ is a G_H-module-epimorphism with $\text{ker.}(f) = A/B$, so the proof follows from (I).

Definition 2.25. An R-G_H-module M is said to be Noetherian (or to satisfy ascending chain condition) if for every ascending chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ of submodules of M, there is a natural number n such that $M_k = M_n$ for all $k \geq n$.

Definition 2.26. An R-G_H-module M is said to satisfy maximum condition if every non-empty collection of submodules of M has a maximum element relative to the ordering of set inclusion.

Proposition 2.27. Let R be a G_H-ring with an i-set and M be a unitary R-G_H-module with condition $(\mathcal{M})$. Then the following statements are equivalent:

(i) M is Noetherian.

(ii) M satisfies maximum condition.

(iii) Every submodule of M is finitely generated.

Proof. (i) $\Rightarrow$ (ii). Straightforward.

(ii) $\Rightarrow$ (iii). Suppose N be a submodule of the G_H-module M. Let S_N be the collection of all finitely generated submodules of N. Clearly, S_N is a non-empty collection, since N contains the zero submodule $<0_M> = \{0_M\}$ of M. Then, S_N has a maximal element; let it be N'. Then, for any $a \in N$, the submodule $N' + \overline{R \circ a}$ of M is contained in N and is finitely generated by the finite set $X \cup \{a\}$, where $N' = <X>$ (since M is a unitary R-G_H-module with condition $(\mathcal{M})$). So $N' + \overline{R \circ a} \in S_{N'}$. But, $N' \subseteq N' + \overline{R \circ a}$ and N' is maximal in S_N. Hence, $N' = N' + \overline{R \circ a}$. Thus, $\overline{R \circ a} \subseteq N'$. Here, $a \in \overline{R \circ a}$ so, $a \in N'$. Hence, $N \subseteq N'$, whence $N = N'$, i.e., N is finitely generated.

(iii) $\Rightarrow$ (i). Let $M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots$ be an ascending chain of submodules of the G_H-module M. Then, $N = \cup_{i \in \mathrm{N}_0} M_i$ is a subgroup of the additive group M. Now, for any $a \in N$, $\exists i \in \mathrm{N}_0$ such that, $a \in M_i$ and thus for any $r \in R$, $r \circ a \subseteq M_i$, whence $r \circ a \subseteq N$. So, N is a submodule of the G_H-module M and hence N is finitely generated by $A = \{a_1, a_2, \ldots, a_n\}$ say. Then, $a_r \in M_{i_r}$ for each $r = 1, 2, \ldots, n$. Let $k = max.\{i_r\}$. Then, $N = <A> \subseteq M_k$, whence, $M_i = M_k$, $\forall i \geq k$.

Proposition 2.28. If M be a Noetherian R-G_H-module, then every submodule of M and every quotient G_H-module of M are also Noetherian.

Proof. The statement concerning submodules is obvious, since the submodules of submodules of a G_H-module M are also submodules of M. That every quotient G_H-module of M is Noetherian, when M is so, is an immediate consequence of the correspondence theorem of G_H-module.

Proposition 2.29. If M be an R-G_H-module and if N be a submodule of M such that N and M/N both are Noetherian, then M is also Noetherian.

Proof. It follows from the correspondence theorem and the fact that the intersection of two submodules of a G_H-module M is also a submodule of M.

3. A CURIOUS EXAMPLE OF G_H-MODULE

Given any non-empty set X, we shall construct in this section a hyperunitary (resp. unitary) G_H-module $\mathcal{F}$ with condition $(\mathcal{M})$, over a G_H-ring R with hyperidentity (resp. with i-set) and with condition $(\mathcal{R})$. Also we shall construct a mapping $\lambda : X \to \mathcal{F}$ such that λ embeds X into $\mathcal{F}$ and $Im(\lambda)$ becomes a spanning set of the G_H-module $\mathcal{F}$ in case when the G_H-ring R has a hyperidentity.

Example 3.1. Let R be a G_H-ring with hyperidentity e and X be any non-empty set. Suppose $\mathcal{F}$ be the set of al those mappings $\theta : X \to R$ such that $\theta(x) = 0_R$ for all but finitely many $x \in X$. On $\mathcal{F}$ we define, $(\theta + \phi)(x) = \theta(x) + \phi(x)$, $\forall\theta, \phi \in \mathcal{F}$ and $\forall x \in X$. Then, $(\mathcal{F}, +)$ is a commutative group with identity $0_{\mathcal{F}}$ defined by $0_{\mathcal{F}}(x) = 0_R$, $\forall x \in X$ and for any $\theta \in \mathcal{F}$, $(-\theta)(x) = -(\theta(x))$, $\forall x \in X$. The additive commutative group $\mathcal{F}$ is a hyperunitary R-G_H-module with condition $(\mathcal{M})$ where the hyperaction o of R on $\mathcal{F}$ being defined by $r \circ \theta = \{\psi \in \mathcal{F} : \psi(x) \in r \circ \theta(x), \forall x \in X\}$, for all $r \in R$ and $\theta \in \mathcal{F}$.

In fact, for any $\theta, \phi \in \mathcal{F}$ and $r, s \in R$, we have the following:

(1) If $\theta = 0_{\mathcal{F}}$, then $r \circ 0_{\mathcal{F}} = \{\psi : \psi(x) \in r \circ 0_{\mathcal{F}}(x) = \{0_R\}\} = \{0_{\mathcal{F}}\}$. So, $r \circ \theta$ is non-empty when $\theta = 0_{\mathcal{F}}$. If $\theta \neq 0_{\mathcal{F}}$, let $x_1, x_2, x_3, \ldots, x_n \in X$ such that $\theta(x_i) \neq 0$ and $\theta(x) = 0$, $\forall x \in X \setminus \{x_1, x_2, x_3, \ldots, x_n\}$. Now, we choose $y_i \in r \circ \theta(x_i)$ for each x_i and then define a mapping $\psi : X \to R$ by stating that $\psi(x_i) = y_i \forall i$ and $\psi(x) = 0 \forall x \in X \setminus \{x_1, x_2, x_3, \ldots, x_n\}$. Then, $\psi \in \mathcal{F}$ and $\psi \in r \circ \theta$ implying that $r \circ \theta$ is non-empty. So, o is an extarnal hypercomposition of $\mathcal{F}$ by R.

(2) $\psi \in (r \circ s) \circ \theta \Rightarrow \psi \in t \circ \theta$ for some $t \in r \circ s \Rightarrow \psi(x) \in t \circ \theta(x) (\forall x \in X) \subseteq (r \circ s) \circ \theta(x) = r \circ (s \circ \theta(x)) \Rightarrow \psi(x) \in r \circ p_x$ for some $p_x \in s \circ \theta(x)$. We define a mapping $\lambda : X \to R$ by stating that $\lambda(x) = p_x$. Since $\theta(x) = 0$ for all but finitely many $x \in X$, so $\lambda \in \mathcal{F}$. So, $\psi(x) \in r \circ \lambda(x)$ for some $\lambda(x) \in s \circ \theta(x) \Rightarrow \psi \in r \circ \lambda$ for some $\lambda \in s \circ \theta \Rightarrow \psi \in r \circ (s \circ \theta)$. Thus, $(r \circ s) \circ \theta \subseteq r \circ (s \circ \theta)$. Similarly $r \circ (s \circ \theta) \subseteq (r \circ s) \circ \theta$, whence $(r \circ s) \circ \theta = r \circ (s \circ \theta)$.

(3) $\psi \in (r + s) \circ \theta \Rightarrow \forall x \in X$, $\psi(x) \in (r + s) \circ \theta(x) \subseteq r \circ \theta(x) + s \circ \theta(x) \Rightarrow \psi(x) = p_x + q_x$ for some $p_x \in r \circ \theta(x)$ and $q_x \in s \circ \theta(x)$. We define $\psi_1(x) = p_x$ and $\psi_2(x) = q_x$. Since, $\theta \in \mathcal{F}$ we have $\psi_1, \psi_2 \in \mathcal{F}$. Moreover, $\psi_1(x) \in r \circ \theta(x)$ and $\psi_2(x) \in s \circ \theta(x)$ for all $x \in X$. Thus $\psi_1 \in r \circ \theta$ and $\psi_2 \in s \circ \theta$. Now for all $x \in X$, $\psi(x) = \psi_1(x) + \psi_2(x) \Rightarrow \psi = \psi_1 + \psi_2 \Rightarrow \psi \in r \circ \theta + s \circ \theta$. So, $(r + s) \circ \theta \subseteq r \circ \theta + s \circ \theta$.

(4) $\psi \in r \circ (\theta + \phi) \Rightarrow \forall x \in X$, $\psi(x) \in r \circ (\theta + \phi)(x) = r \circ (\theta(x) + \phi(x)) \subseteq r \circ \theta(x) + r \circ \phi(x)$.

Thus by the arguement same as that applied in (3), we can show that $\psi \in ro\theta + ro\phi$. So, $ro(\theta + \phi) \subseteq ro\theta + ro\phi$.

(5) (a) Let the G_H-ring R has a hyperidentity e. Then for each $x \in X$, $\theta(x) \in eo\theta(x)$ (since, $\theta(x) \in R$). Hence, by definition, $\theta \in e \text{ o } \theta$. Thus $\mathcal{F}$ is a hyperunitary R-G_H-module.

(b) Let the G_H-ring R has an i-set $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$. Then for each $x \in X$, $\theta(x) \in \sum_{i=1}^{n} e_i \circ \theta(x) \Rightarrow \theta(x) = \sum_{i=1}^{n} p_x^i$ for some $p_x^i \in e_i o\theta(x)$, $i = 1, 2, \ldots, n$. We define for each i, a mapping ψ_i such that $\forall x \in X$, $\psi_i(x) = p_x^i$. Since $\theta(x) = 0$ for all but finitely many $x \in X$, so $\psi_i \in \mathcal{F}$. Moreover, for each $x \in X$, $\psi_i(x) = p_x^i \in e_i o\theta(x) \Rightarrow \psi_i \in e_i o\theta$ (for each i). Now, $\theta(x) = \sum_{i=1}^{n} p_x^i = \sum_{i=1}^{n} \psi_i(x) = \left(\sum_{i=1}^{n} \psi_i\right)(x) \Rightarrow \theta = \sum_{i=1}^{n} \psi_i \in \sum_{i=1}^{n} e_i \circ \theta$. Thus $\mathcal{F}$ is a unitary R-G_H-module.

(6) $\psi \in (-r) \text{ o } \theta \Rightarrow \forall x \in X$, $\psi(x) \in (-r) \text{ o } \theta(x) = -(r \text{ o } \theta(x))$ (since, G_H-ring R satisfies condition $(\mathcal{R})$) $\Rightarrow (-\psi)(x)) \in r \text{ o } \theta(x) \Rightarrow -\psi \in r \text{ o } \theta \Rightarrow \psi \in -(r \text{ o } \theta)$. So, $(-r) \text{ o } \theta \subseteq -(r \text{ o } \theta)$. Similarly $-(r \text{ o } \theta) \subseteq (-r) \text{ o } \theta$ whence $(-r) \text{ o } \theta = -(r \text{ o } \theta)$. By the similar arguement, $r \text{ o } (-\theta) = -(r \text{ o } \theta)$. Hence, the G_H-module $\mathcal{F}$ is a R-G_H-module with condition $(\mathcal{M})$.

Now, we define a mapping $\lambda : X \to \mathcal{F}$ by stating that $\forall x \in X$, $x \to \lambda(x) = \lambda_x \in \mathcal{F}$ such that

$$\lambda_x(y) = \begin{cases} e & \text{when } y = x \\ 0_R & \text{when } y \neq x \end{cases}$$

when the G_H-ring R has been taken to have a hyperidentity e.

This is to be observed that for any $x, y \in X$, $\lambda_x = \lambda_y \Rightarrow \lambda_x(y) = \lambda_y(y) \Rightarrow \lambda_x(y) = e \Rightarrow x = y$, whence λ is an one to one correspondence between X and $I_m(\lambda)$. Let $0_{\mathcal{F}} \neq \theta \in \mathcal{F}$. Then, there are $x_1, x_2, \ldots, x_n \in X$ such that $\theta(x_i) \neq 0_R$ for each i, and $\theta(x) = 0_R$, $\forall x \in X \backslash \{x_i\}$. Now, for each i we define $\theta_i \in \mathcal{F}$ by stating that $\theta_i(x_i) = \theta(x_i)$ and $\theta_i(x) = 0_R$, $\forall x \in X$ with $x \neq x_i$. Then, $\hat{\theta}_i \in \theta(x_i) o \lambda_{x_i}$. In fact, $\theta_i(x_i) = \theta(x_i) \in \theta(x_i) o\ e = \theta(x_i) \text{ o } \lambda_{x_i}(x_i)$ and for any $x \neq x_i$, $\theta_i(x) = 0_R \in \theta(x) \text{ o } 0_R = \theta(x) \text{ o } \lambda_{x_i}(x)$. Clearly then, $\theta = \sum_{i=1}^{n} \theta_i \in \sum_{i=1}^{n} \left(\theta(x_i) \circ \lambda_{x_i}\right)$. Again, $\{0_{\mathcal{F}}\} = 0_{\mathcal{F}}(x) o \lambda_x$, $\forall x \in X$. Hence, $Im(\lambda) = \{\lambda_x : x \in X\}$ spans the G_H-module $\mathcal{F}$.

Now suppose that the G_H-ring R has an i-set $\mathcal{E} = \{e_1, e_2, \ldots, e_n\}$. Then for each i, we define a mapping $\lambda^{e_i} : X \to \mathcal{F}$ by stating that $\forall x \in X$, $x \to \lambda^{e_i}(x) = \lambda^{e_i}_x \in \mathcal{F}$ such that

$$\lambda^{e_i}_x(y) = \begin{cases} e_i & \text{when } y = x \\ 0_R & \text{when } y \neq x \end{cases}$$

Let $\lambda : X \to \mathcal{F}$ defined by $x \to \lambda(x) = \lambda_x = \sum_{i=1}^{n} \lambda^{e_i}_x \in \mathcal{F}$. This is to be noted that the mapping λ so constructed does not embed X onto $Im(\lambda)$.

We now propose the following definition of *free G_H-module.*

Definition 3.2. Let R be a G_H-ring with a hyperidentity and with condition $(\mathcal{R})$ and let X be a non-empty set. By a *free R-G_H-module on X,* we mean a hyperunitary R-G_H-module F with condition $(\mathcal{M})$, together with a mapping $f : X \to F$ such that, for every hyperunitary R-G_H-module M with condition $(\mathcal{M})$ and every mapping $g : X \to M$, there is a unique R-G_H-module homomorphism $h : F \to M$ such that $h \circ f = g$. We denote such an R-G_H-module by (F, f).

We conclude this paper by keeping the following three subsequent problems open.

Problem 3.3. Let R be G_H-ring with a hyperidentity and satisfies condition $(\mathcal{R})$. If for any non-empty set X, $\mathcal{F}$ be the hyperunitary R-G_H-module with condition $(\mathcal{M})$, so constructed in example 3.1 and $\lambda : X \to \mathcal{F}$ be the mapping described in example 3.1, then can $(\mathcal{F}, \lambda)$ be a free R-G_H-module on X, in the sense of the proposed definition (3.2) of freeness of G_H-module?

Problem 3.4. If $(\mathcal{F}, \lambda)$ comes to be a free R-G_H-module and (F, f) be another free R-G_H-module, can then G_H-module F be isomorphic to the G_H-module $\mathcal{F}$?

Problem 3.5. Let R be a G_H-ring with an i-set and with condition $(\mathcal{R})$. Suppose $\mathcal{F}$ be the unitary R-G_H-module satisfying condition $(\mathcal{M})$, as defined in the example 3.1. Then does there exist a mapping $\lambda : X \to \mathcal{F}$ such that it embeds X into $\mathcal{F}$ and $Im(\lambda)$ becomes a spanning set of the G_H-module $\mathcal{F}$?

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REFERENCES

1. Chaopraknoi, S. and Kemprasit, Y., Linear Transformation Semigroups admitting the structure of a Semihyperring with Zero; Italian Journal of Pure and Applied Mathematics; 17, January 2005, 213-222.
2. Corsini, P., Prolegomena of Hypergroup Theory; Aviani Editore, Italy, 1993.
3. Fotea, V. L., Fuzzy Hypermodules; Computers and Mathematics with Applications, 57(3), 2009, 466-475.
4. Krasner, M., A Class of Hyperring and Hyperfield; Internat. J. Math. and Math. Sci., 6(2), 1983, 307-312.
5. Massouros, Ch. G., Free and Cyclic Hypermodules; Ann. Mat. Pura ed Appl., Serie IV, Tomo CL, (1988).
6. R. Rota, Sugli Iperanelli Moltiplicatie, Rend. di Mat. VII(4) (1982) 711-724. (1999) 485-497.
7. Sen, M. K., Ameri, R. and Chowdhury, G., Fuzzy hypersemigroups, Soft Computing, 2007, DOI 10.1007/s00500-007-0257-9.
8. Sen M. K. and Dasgupta, Utpal, Some Aspects of G_H-rings, Analele Stintifice Ale Universitatii 'Al. I. Cuza', IASI, Sectiunea Matematica, LVI, 2010, 253-272.
9. Sen, M. K. and Dasgupta, Utpal, Hypersemirings; Bulletin of Calcutta Mathematical Society; 100(2), 2008, 143-156.

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A CHAIN OF RESULTS ON FIXED POINTS IN AN ε-CHAINABLE METRIC SPACE

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ABSTRACT : In this paper we present some common fixed point theorems for an ε-chainable metric space.

Key words : Metric space, ε-chainable metric space, Chainable metric space and Fixed point theorem.
Mathematics Subject Classification. 47H10.

1. INTRODUCTION

In 1883 Cantor defined connectedness with the help of ε-chains which have been studied extensively by many mathematicians [Reference 1, 2, 3, 4, 5, 6, 7]. These ε-chains were defined between two given points and comprised of points of the metric space under consideration.

On the other hand a number of fixed-point theorems have been proved for metric spaces, 2-metric spaces, Hilbert spaces, Banach spaces [Reference 8, 9, 10]. The object of this paper is to obtain some common fixed-point theorems for an ε-chainable metric space using general contractive conditions.

2. PRELIMINARIES

Definition 2.1. Let (X, d) be a metric space. An ε-chain is a finite succession of points a_0, a_1, a_2, a_3,..., a_{n-1}, a_n in X such that $d(a_{i-1}, a_i) \geq \varepsilon$ *for* $i = 1, 2, ..., n$. The integer n is called the length of the ε-chain.

Definition 2.2. A space (X, d) is ε-chainable (ε-connected) if every pair of points in it can be joined by an ε-chain of points in the set X and (X, d) is called chainable if it is ε-chainable for each positve ε.

The space (X, d) is called complete chainable metric space if every Cauchy sequence converges in it.

Throughout this paper length of ε-chain between any two points x and y in X means length of shortest ε-chain between points x and y in X.

Example 2.2.(a) $Y = A \cup B$ and d be the usual metric on Y

where

$A = \{1, 1/2, 1/3\}$ *and* $B = \{1/2^2, 1/3^2, 1/4^2\}$

Then Y is ε-chainable for $\varepsilon = 1/2$ and length of biggest *ε-chain in* Y *is* 2.

Example 2.2. (b) Let d be the usual metric on $R \times R$ then the set $C_0 \cup C_1 \cup C_2 \ldots\ldots \cup C_n$ is ε-chainable where $C_0, C_1, \ldots, C_n = C$ are circumferences of concentric circles of radi R/n, $2R/n$, $3R/n$, ..., nR/n respectively, here $\varepsilon \geq R/n$ and the length of biggest ε-chain in space $((C_0 \cup C_1 \cup C_2 \cup \ldots \cup C_n), d)$ is n.

Example 2.2. (c) Let (R, d) be the usual metric space and $Y = A \cup B$ where $A = [1, 2)$ *and* $B = (2, 3]$ Then (Y, d) is a disconnected subspace of (R, d) but is ε-chainable for every $\varepsilon > 0$. If $\varepsilon = 1/2$ then length of biggest ε-chain in space (Y, d) is 5.

3. MAIN RESULT

Theorem 3.1. Let (X, d) be a complete ε-chainable metric space and let S, T be two self maps of X, if

$$\varepsilon = d(Sx, Ty) \leq a_1 d(x, Sx) + a_2 d(y, Ty) + a_3 d(x, Ty) + a_4 d(y, Sx) + a_3 d(x, y) \quad \ldots(1)$$

for all $x, y \in X$ where a_1, a_2, a_3, a_4, a_5 are non negative reals. And

$$\text{Max } \{(M(a_1 + a_2 + 2a_4 + a_5), (a_2 + a_3 + a_4 + a_5)\} < 1 \quad \ldots (2)$$

where M is the length of the biggest ε-chain in space (X, d) such that $M \geq 0$. Then S and T have unique common fixed point.

Proof. We define a sequence $\{x_n\}$ as follows

$Sx_1 = x_0$, $Sx_3 = x_2$, $Sx_5 = x_4$, ..., $Sx_{2n+1} = x_{2n}$

$Tx_2 = x_1$, $Tx_4 = x_3$, $Tx_6 = x_5$, ..., $Tx_{2n+2} = x_{2n+1}$

for $n = 0, 1, 2, \ldots$..

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we see that x_{2n} is a fixed point of S and T therefore we suppose that no two consecutie terms of sequence $\{x_n\}$ are equal. If N is the length of ε-chain between x_{2n+1} and x_{2n+2} then ε-chainability of (X, d) gives.

$$d(x_{2n+1}, x_{2n+2}) \leq N\varepsilon$$

$$\leq M\varepsilon \qquad [as\ M \geq N]$$

$$= M\, d(Sx_{2n+1}, Tx_{2n+2}) \text{ [using (1)]}$$

$$\leq M[a_1 d(x_{2n+1}, Sx_{2n+1}) + a_2 d(x_{2n+2}, Tx_{2n+2}) + a_3 d(x_{2n+1}, Tx_{2n+2}) + a_4 d(x_{2n+2}, Sx_{2n+1}) + a_5 d(x_{2n+1}, x_{2n+2})] \text{ [using (1)]}$$

$$= M[a_1 d(x_{2n+1}, x_{2n}) + a_2 d(x_{2n+2}, x_{2n+1}) + a_3 d(x_{2n+1}, x_{2n+1}) + a_4 d(x_{2n+2}, x_{2n}) + a_5 d(x_{2n+1}, x_{2n+2})]$$

$$\leq M[a_1 d(x_{2n+1}, x_{2n}) + a_2 d(x_{2n+2}, x_{2n+1}) + a_4 d(x_{2n+2}, x_{2n+1}) + a_4 d(x_{2n+1}, x_{2n}) + a_5 d(x_{2n+1}, x_{2n+2})]$$

$$[1 - M(a_2 + a_4 + a_5)]d(x_{2n+1}, x_{2n+2}) \leq M(a_1 + a_4)d(x_{2n}, x_{2n+1})$$

$$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{M(a_1 + a_4)}{[1 - M(a_2 + a_4 + a_3)]} d(x_{2n}, x_{2n+1})$$

In general

$$\Rightarrow d(x_n, x_{n+1}) \leq \left[\frac{M(a_1 + a_4)}{[1 - M(a_2 + a_4 + a_5)]}\right] d(x_{n-1}, x_n)$$

$$\Rightarrow d(x_n, x_{n+1}) \leq k\, d(x_{n-1}, x_n) \text{ Where } k = \left[\frac{M(a_1 + a_4)}{[1 - M(a_2 + a_4 + a_5)]}\right] < 1 \text{ [using (2)]}$$

$$\Rightarrow d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

Now we prove that $\{x_n\}$ is a Cauchy sequence, if p is any positie integer then we have.

$$d(x_n, x_{n+p}) \leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \dots + d(x_{n+p-1}, x_{n+p})$$

$$\leq \{k^n + k^{n+1} + k^{n+2} + \dots + k^{n+p-1}\} d(x_0, x_1)$$

$$\leq \frac{k^n}{(1-k)} d(x_0, x_1)$$

as $n \to \infty$ $d(x_n, x_{n+p}) \to 0$

Therefore $\{x_n\}$ is a Cauchy sequence in X. As X is a complete space so there exists a point x in X such that $\lim_{n \to \infty} x_n = x$.

Existence of fixed point:

Consider

$$\begin{aligned} d(x, Tx) &\leq d(x, x_{2n}) + d(x_{2n}, Tx) \\ &= d(x, x_{2n}) + d(Sx_{2n+1}, Tx) \\ &\leq d(x, x_{2n}) + a_1 d(x_{2n+1}, Sx_{2n+1}) + a_2 d(x, Tx) + a_3 d(x_{2n+1}, Tx) + a_4 d(x, Sx_{2n+1}) \\ &\qquad + a_5 d(x_{2n+1}, x) \\ &= d(x, x_{2n}) + a_1 d(x_{2n+1}, x_{2n}) + a_2 d(x, Tx) + a_3 d(x_{2n+1}, Tx) + a_4 d(x, x_{2n}) \\ &\qquad + a_5 d(x_{2n+1}, x) \end{aligned}$$

As $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are subsequences of $\{x_n\}$, *as* $n \to \infty$ $x_{2n} \to x$ *and* $x_{2n+1} \to x$

Therefore

$d(x, Tx) \leq (a_2 + a_3) d(x, Tx)$

$\Rightarrow [1 - (a_2 + a_3)] d(x, Tx) \leq 0 \quad$ (*As* $(a_2 + a_3) < 1$, *by* (2))

$\Rightarrow d(x, Tx) = 0$

$\Rightarrow Tx = x$

Similarly we can prove $Sx = x$. This shows that x is common fixed point of S and T.

Uniqueness: Let u be another fixed point of S and T. $d(x, u) = d(Sx, Tu)$

$$\leq a_1 d(x, Sx) + a_2 d(u, Tu) + a_3 d(x, Tu) + a_4 d(u, Sx) + a_5 d(x, u) \text{ By (1)}$$

$\Rightarrow d(x, u) \leq [a_3 + a_4 + a_5] d(x, u)$

$\Rightarrow [1 - (a_3 + a_4 + a_5)] d(x, u) \leq 0$

$\Rightarrow d(x, u) = 0 \qquad$ (*as* $(a_3 + a_4 + a_5) < 1$)

$\Rightarrow x = u$

This Complete the proof.

Theorem 3.2. Let S, T be two self maps of a complete ε-chainable metric space (X, d) satisfying

$$\varepsilon^2 = \{d(Sx, Tu)\} \leq \alpha d(x, Sx) d(y, Ty) + \beta d(x, Ty) d(y, Sx) \qquad \text{... (3)}$$

for all $x, y \in X$ where α, β are non negative reals. And $0 \leq M^2\alpha < 1$ ... (4)

where M is the length of the biggest ε-chain in X such that $M \geq 0$, then S, T have a common fixed point. If further $\beta < 1$ then S and T have a unique fixed point in X.

Proof. We define a sequence $\{x_n\}$ as follows

$$Sx_1 = x_0,\ Sx_3 = x_2,\ Sx_5 = x_4,\ \ldots,\ Sx_{2n+1} = x_{2n}$$
$$Tx_2 = x_1,\ Tx_4 = x_3,\ Tx_6 = x_5,\ \ldots,\ Tx_{2n+2} = x_{2n+1}$$
$$\textit{for } n = 0, 1, 2 \ldots$$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we see that x_{2n} is a fixed point of S and T. Therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal. If N is the length of ε-chain between x_{2n+1} and x_{2n+2} then ε-chainability of (X, d) gives.

$$d(x_{2n+1}, x_{2n+2}) \leq N\,\varepsilon$$
$$\{d(x_{2n+1}, x_{2n+2})\}^2 \leq N^2\,\varepsilon^2$$
$$\leq M^2\varepsilon^2\ [as\ M^2 \geq N^2]$$
$$= M^2\{d(Sx_{2n+1}, Tx_{2n+2})\}^2\ [using\ (3)]$$
$$\leq M^2\alpha d(x_{2n+1}, Sx_{2n+1})d(x_{2n+2}, Tx_{2n+2}) + M^2\beta d(x_{2n+1}, Tx_{2n+2})d(x_{2n+2}, Sx_{2n+1})$$

By (3)

$$= M^2\alpha d(x_{2n+1}, x_{2n})d(x_{2n+2}, x_{2n+1}) + M^2\beta d(x_{2n+1})d(x_{2n+2}, x_{2n})$$
$$\Rightarrow \{d(x_{2n+1}, x_{2n+2})\}^2 \leq M^2\alpha d(x_{2n+1}, x_{2n})d(x_{2n+2}, x_{2n+1})$$
$$\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq M^2\alpha d(x_{2n}, x_{2n+1})$$

In general

$$\Rightarrow d(x_n, x_{n+1}) \leq M^2\alpha d(x_{n-1}, x_n)$$
$$\Rightarrow d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)\ where\ k = M^2\,\alpha < 1\ By\ (4)$$
$$\Rightarrow d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

Now we can prove that $\{x_n\}$ is a Cauchy sequence (as proved in theorem (3.1)).

As (X, d) is a complete space so there exists a point x in X such that $\lim_{n\to\infty} x_n = x$.

Existence of fixed point:

Consider

$$d(x, Tx) \leq d(x, x_{2n}) + d(x_{2n}, Tx)$$
$$\Rightarrow d^2(x, Tx) \leq d^2(x, x_{2n}) + d^2(x_{2n}, Tx) + 2d(x, x_{2n})d(x_{2n}, Tx)$$

$$
\begin{aligned}
&= d^2(x, x_{2n}) + d^2(Sx_{2n+1}, Tx) + 2d(x, x_{2n})d(x_{2n}, Tx) \\
&\leq d^2(x, x_{2n}) + \alpha d(x_{2n+1}, Sx_{2n+1})d(x, Tx) + \beta d(x_{2n+1}, Tx)d(x, Sx_{2n+1}) \\
&\qquad + 2d(x, x_{2n})d(x_{2n}, Tx) \\
&= d^2(x, x_{2n}) + \alpha d(x_{2n+1}, x_{2n})d(x, Tx) + \beta d(x_{2n+1}, Tx)d(x, x_{2n}) \\
&\qquad + 2d(x, x_{2n})d(x_{2n}, Tx)
\end{aligned}
$$

As $\{x_{2n+1}\}$ and $\{x_{2n}\}$ are subsequences of $\{x_n\}$, *as* $n \to \infty$ $x_{2n} \to x$ *and* $x_{2n+1} \to x$

Therefore

$d(x, Tx) \leq 0$

$\Rightarrow Tx = x$

Similarly we can prove $Sx = x$. This shows that x is common fixed point of S and T.

Uniqueness.

Let u be another fixed point of S and T.

$d(x, u) \leq d(Sx, Tu)$

$$
\begin{aligned}
\Rightarrow d^2(x, u) &\leq d^2(Sx, Tu) \\
&\leq \alpha d(x, Sx)d(u, Tu) + \beta d(x, Tu)d(u, Sx) \text{ By (3)} \\
&\leq \beta d^2(x, u)
\end{aligned}
$$

$\Rightarrow (1 - \beta)d^2(x, u) \leq 0$

$\Rightarrow d(x, u) = 0$ (as $\beta < 1$)

$\Rightarrow x = u$

This complete the proof.

Theorem 3.3. Let S, T be two self maps of a complete ε-chainable metric space (X, d) satisfying

$$\varepsilon = d(Sx, Ty) \leq c\left[\frac{\{d(y, Sx)\}^2 + \{d(x, Ty)\}^2}{d(y, Sx) + d(x, Ty)}\right] \quad \text{..... (5)}$$

for all x, y in X such that $d(y, Sx) + d(x, Ty) \neq 0$. *And* $0 \leq cM < ½$, $c < 1$ ..... (6) where M is the length of the biggest ε-chain in X such that $M \geq 0$. Then S, T have a common unique fixed point in X.

Proof. We define a sequence $\{x_n\}$ as follows

$$Sx_1 = x_0,\ Sx_3 = x_2,\ Sx_5 = x_4,\ \ldots..,\ Sx_{2n+1} = x_{2n}$$

$$Tx_2 = x_1,\ Tx_4 = x_3,\ Tx_6 = x_5,\ \ldots..,\ Tx_{2n+2} = x_{2n+1}$$

for $n = 0, 1, 2, \ldots..$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we see that x_{2n} is a fixed point of S and T therefore we suppose that no two consecutive terms of sequence $\{x_n\}$ are equal. If N is the length of ε-chain between x_{2n+1} and x_{2n+2} then ε-chainability of (X, d) gives.

$$d(x_{2n+1}, x_{2n+2}) \qquad \le N\,\varepsilon$$

$$\le M\,\varepsilon\ [as\ M \ge N]$$

$$= M\,d(Sx_{2n+1}, Tx_{2n+2})\quad [By\ (5)]$$

$$\le Mc\left[\frac{\{d(x_{2n+2}, Sx_{2n+1})\}^2 + \{d(x_{2n+1}, Tx_{2n+2})\}^2}{d(x_{2n+2}, Sx_{2n+1}) + d(x_{2n+1}, Tx_{2n+2})}\right]\quad [By\ (5)]$$

$$= Mc\left[\frac{\{d(x_{2n+2}, x_{2n})\}^2 + \{d(x_{2n+1}, x_{2n+1})\}^2}{d(x_{2n+2}; x_{2n}) + d(x_{2n+1}, x_{2n+1})}\right]$$

$$= Mc[d(x_{2n+2}, x_{2n})]$$

$$\le Mc[d(x_{2n+2}, x_{2n+1}) + d(x_{2n+1}, x_{2n})]$$

$$\Rightarrow [1 - Mc]d(x_{2n+2}, x_{2n+1}) \le Mcd(x_{2n+1}, x_{2n})$$

$$\Rightarrow d(x_{2n+1}, x_{2n+2}) \le \frac{Mc}{[1 - Mc]} d(x_{2n}, x_{2n+1})$$

In general

$$d(x_n, x_{n+1}) \le \frac{Mc}{[1 - Mc]} d(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \le kd(x_{n-1}, x_n)\ Where\ k = \frac{Mc}{[1 - Mc]} < 1\ [By\ (6)]$$

$$\Rightarrow d(x_n, x_{n+1}) \le k^n d(x_0, x_1)$$

Now we can prove that $\{x_n\}$ is a Cauchy sequence (as proved in theorem (3.1).

As (X, d) is a complete space so there exists a point x in X such that $\lim_{n\to\infty} x_n = x$.

Existence of fixed point:

$$\begin{aligned} d(x, Tx) &\le d(x, x_{2n}) + d(x_{2n}, Tx) \\ &= d(x, x_{2n}) + d(Sx_{2n+1}, Tx) \\ &\le d(x, x_{2n}) + c\left[\frac{\{d(x, Sx_{2n+1})\}^2 + \{d(x_{2n+1}, Tx)\}^2}{d(x, Sx_{2n+1}) + d(x_{2n+1}, Tx)}\right] \quad [By\ (5)] \\ &= d(x, x_{2n}) + c\left[\frac{\{d(x, x_{2n})\}^2 + \{d(x_{2n+1}, Tx)\}^2}{d(x, x_{2n}) + d(x_{2n+1}, Tx)}\right] \end{aligned}$$

As $\{x_{2n}\}$ *and* $\{x_{2n+1}\}$ *are subsequences of* $\{x_n\}$ *as* $n \to \infty$ $x_{2n} \to x$ *and* $x_{2n+1} \to x$.

$$\Rightarrow d(x, Tx) \le c\ d(x, Tx)$$

$$\Rightarrow [1 - c]d(x, Tx) \le 0 \quad (as\ c < 1)$$

$\Rightarrow Tx = x$ *similarly we can prove* $Sx = x$.

This shows that x is a common fixed point of S and T.

Uniqueness: Let u be another fixed point of S and T.

$$\begin{aligned} d(x, u) &= d(Sx, Tu) \\ &\le c\left[\frac{\{d(u, Sx)\}^2 + \{d(x, Tu)\}^2}{d(u, Sx) + d(x, Tu)}\right] \quad [By\ (5)] \end{aligned}$$

$$d(x, u)[d(u, x) + d(x, u) \le c[\{d(u, x)\}^2 + \{d(x, u)\}^2]$$

$$\Rightarrow 2\{d(x, u)\}^2 \le 2c\{d(x, u)\}^2$$

$$\Rightarrow [1 - c]\{d(x, u)\}^2 \le 0$$

$$\Rightarrow d(x, u) = 0 \quad (as\ c < 1)$$

$$\Rightarrow u = x$$

This complete the proof.

Theorem 3.4. Let S, T be two self maps of a complete ε-chainable metric space (X, d) satisfying

$$\varepsilon = d(Sx, Ty) \leq \delta\max\{d(x, y), [d(x, Sx) + d(y, Ty)], [d(x, Ty) + d(y, Sx)]\} \quad ...(7)$$

for all x, y *in* X *and* $0 \leq \delta M < \frac{1}{2}$, $\delta < \frac{1}{2}$ ..... (8) where M is the length of the biggest ε-chian in X such that $M \geq 0$. Then S, T have a common unique fixed point in X.

Proof. We define a sequence $\{x_n\}$ as follows

$$Sx_1 = x_0,\ Sx_3 = x_2,\ Sx_5 = x_4,\ .....,\ Sx_{2n+1} = x_{2n}$$

$$Tx_2 = x_1,\ Tx_4 = x_3,\ Tx_6 = x_5,\ .....,\ Tx_{2n+2} = x_{2n+1}$$

for $n = 0, 1, 2, ...$

If $x_{2n} = x_{2n+1} = x_{2n+2}$ for some n, then we see that x_{2n} is a fixed point of S and T therefore we suppose that no two consecutie terms of sequence $\{x_n\}$ are equal. If N is the length of ε-chain between x_{2n+1} and x_{2n+2} then ε-chainability of (X, d) gives.

$$d(x_{2n+1}, x_{2n+2}) \leq N\varepsilon \qquad \leq N\varepsilon$$

$$\begin{aligned}
&\leq M\varepsilon \quad [as\ M \geq N] \\
&= Md(Sx_{2n+1}, Tx_{2n+2}) \\
&\leq M\delta max\{d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, Sx_{2n+1}) + d(x_{2n+2}, Tx_{2n+2}), \\
&\qquad d(x_{2n+1}, Tx_{2n+2}) + d(x_{2n+2}, Sx_{2n+1})\} \\
&= M\delta max\{d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1}), \\
&\qquad d(x_{2n+1}, x_{2n+1}) + d(x_{2n+2}, x_{2n})\} \\
&= M\delta max\{d(x_{2n+1}, x_{2n+2}), d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1}), d(x_{2n+2}, x_{2n})\} \\
&= \delta M[d(x_{2n+1}, x_{2n}) + d(x_{2n+2}, x_{2n+1})] \\
&\Rightarrow [1 - \delta M]d(x_{2n+1}, x_{2n+2}) \leq \delta Md(x_{2n}, x_{2n+1}) \\
&\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq \frac{\delta M}{[1 - \delta M]} d\ (x_{2n}, x_{2n+1}) \\
&\Rightarrow d(x_{2n+1}, x_{2n+2}) \leq kd(x_{2n}, x_{2n+1})\ where\ k = \frac{\delta M}{[1 - \delta M]} < 1\ [By\ (8)]
\end{aligned}$$

In general

$$d(x_n, x_{n+1}) \leq kd(x_{n-1}, x_n)$$

$$d(x_n, x_{n+1}) \leq k^n d(x_0, x_1)$$

Now we can prove that $\{x_n\}$ isa Cauchy sequence (as proved in theorem (3.1)).

As (X, d) is a complete space so there exists a point x in X such that $\lim\limits_{n\to\infty} x_n = x$.

Existence of fixed point:

Consider

$$d(x, Tx) \leq d(x, x_{2n}) + d(x_{2n}, Tx)$$

$$= d(x, x_{2n}) + d(Sx_{2n+1}, Tx)$$

$$\leq d(x, x_{2n}) + \delta Max\{d(x_{2n+1}, x), d(x_{2n+1}, Sx_{2n+1}) + d(x, Tx), d(x_{2n+1}, Tx) + d(x, Sx_{2n+1})\}$$

$$\leq d(x, x_{2n}) + \delta Max\{d(x_{2n+1}, x), d(x_{2n+1}, x_{2n}) + d(x, Tx), d(x_{2n+1}, Tx) + d(x, x_{2n})\}$$

As $\{x_{2n+1}\}$ *and* $\{x_{2n}\}$ *are subsequences of* $\{x_n\}$ *as* $n \to \infty$ $x_{2n+1} \to x$ *and* $x_{2n} \to x$

Therefore

$$d(x, Tx) \leq d(x, x) + \delta Max\{d(x, x), d(x, x) + d(x, Tx), d(x, Tx) + d(x, x)\}$$

$$\Rightarrow d(x, Tx) \leq \delta d(x, Tx)$$

$$\Rightarrow (1 - \delta)d(x, Tx) \leq 0 \ (as\ \delta < 1/2)$$

$$\Rightarrow d(x, Tx) = 0$$

$$\Rightarrow Tx = x$$

Similarly we can proof $Sx = x$.

This shows that x is the common fixed point of S and T.

Uniqueness:

Let u be another fixed point of S and T.

$$d(x, u) = d(Sx, Tu)$$

$$\leq \delta max\{d(x, u), d(x, Sx) + d(u, Tu), d(x, Tu) + d(u, Sx)]$$

$$d(x, u) \leq 2\delta d(x, u)$$

$\Rightarrow [1 - 2\delta]d(x, u) \leq 0$

$\Rightarrow d(x, u) = 0 \quad [as\ \delta < \tfrac{1}{2}]$

$\Rightarrow x = u$

This complete the proof.

Theorem 3.5. Let S, T be two self maps of a complete ε-chainable metric space (X, d) satisfying

$$d(Sx, Ty) = \varepsilon \leq \delta \max\{d(y, Sx), d(x, Ty), [d(x, Ty) + d(y, Sx)]/2\} \qquad \text{... (9)}$$

for all x, y in X and $0 \leq \delta M < \tfrac{1}{2}$, $\delta < \tfrac{1}{2}$ ..... (10) where M is the length of the biggest ε-chian in X such that $M \geq 0$. Then S, T have a common unique fixed point in X.

Proof. Proof of this theorem is similar to the proof of theorem (3.4).

Theorem 3.6. Let S, T be two self maps of a complete ε- chainable metric space (X, d) satisfying

$$d(Sx, Ty) = \varepsilon \leq \delta\max\{cd(x, y), [d(x, Sx) + d(y, Ty)]/2, [d(x, Ty) + d(y, Sx)]/2\} \qquad \text{...(11)}$$

for all x, y in X with $0 < c < \tfrac{1}{2}$, $0 \leq \delta M < 1$ and $\delta < \tfrac{1}{2}$ ..... (12) and where M is the length of the biggest ε-chian in X such that $M \geq 0$. Then S, T have a common unique fixed point in X.

Proof. Proof of this theorem is similar to the proof of theorem (3.4)

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REFERENCES

1. Beer, G., Which connected metric spaces are compact? Proc. Amer. Math. Soc. Volume 83, Number 4, 1981.
2. Bellamy, D. P., Short paths in homogeneous continua Topology Appl. 26 (1987) 287-291.
3. Bing, R. H., Concerning hereditarily indecomposable continua, pacific J. Math. (1951) 43-51.
4. Newman, M.H.A. Elements of Topology of plane sets of points Cambridge univ. Press, New york, 1961.
5. MINC, P., A fixed point theorem for weakly chainable plane, Continua., transactions of the American. Mathematical Society, Volume 317, Number 1, January 1990.
6. Naimpally S. A and pareek, C.M., on the compactness of connected sets (preprint).

7. Kiran Shrivastava and Geeta Agrawal, Characterization of ε-chainable sets in metric spaces Indian J. Pure aple. Math 33(6) : 933-940 June 2002.

8. Hardy, G.E. and Rogers T.D. "A generalization of a fixed point theorem of Reich canod math. Bull 16 (1973) 201-206.

9. B. Fisher, Fixed point and constant muppings on metric spaces. Atti Accad naz.lincel rend cl.sci.Fis. Mat. Nature (8) 61 (1976) no. 5, 329-332 (1977) MR57 # 10694 zbl 375.54034.

10. B. E. Rhodes, Contractive deffinitions and continuity, contemporary mathematics volume, 72, 1988 (233-245).

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FREEING SETS AND ITS GRAPH THEORETIC RELEVANCE IN A RING R

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ABSTRACT : The results of this paper are in case of a directed—graph in a ring R. The so-called freeing sets and the cliques are playing key roles in our discussion. In some special types of rings with its reduced character without infinite directed-cliques gives rise to some finiteness conditions on right annihilators of the subsets of R. The torsion-free right sub commutative character is an another aspect of importance in our discussiion. Existence of some sort of freeing sets in the ring is shown in case of a torsion-free ring which is a coloring. On the other hand a one sided ideal with freeing sets gives rise to the co-incidence of the clique and the chromatic number of the ideal with its corresponding notions when the ideal is intersected by the right annihilator of the freeing set.

Key words and pharases : Freeing sets, clique, chromatic number, prime sub-module, torsion-free ring, sub-commutativity.

AMS Subject Classification 2000: 16D25, 16D80, 16N40, 16P60, 05C15.

1. INTRODUCTION

In this paper we deal with directed graphs in a ring R and investigate some properties connecting the so-called freeing sets and its *directed-cliques*.

G_R is the graph (V, E) where the elements of the ring R is the set V of vertices and E is the set of edges. an element $x(\in R)$ is *adjacent* to $y(\in R)$ if $x \cdot y = 0$ or we call it "x and y are *adjacent*". i.e. two elements x and y are *adjacent* if there is a directed edge between them (clearly such a graph is a directed one). Two elements x and y are "*adjacent to each other*" if $x \cdot y = 0 = y \cdot x$. We call it *adjacent* simply. If x and y are not distinct then (x, x) is a *loop*. A subset C (finite/infinite) of R is a *clique* if any two elements of C are *adjacent*. A finite subset $C = \{a, b, ..., d\} = \{x_1, x_2, ..., x_n\}$ say, we have $x_i \cdot x_j = 0$ for all $i < j$ and it is to be noted that C will be a clique as in above sense, if C is a *directed-clique* for all possible labeled description of it. Now a countably infinite subset C is a *directed-clique,* if for each $n \in N$, a labeled subset viz. $X = \{a, b, ..., d\} = \{x_1, x_2, ..., x_n\}$ $(\subseteq C)$ is a *directed-clique* and such a countably infinite set is a clique if for each $n \in N$ and for each above type of labeled description of X is a *directed-clique.* The minimum number of colors which can be assigned to the elements of R so that no two *adjacent* elements have same color is the *chromatic* number of R and is denoted by $\chi(R)$. The cardinality of maximal *directed-clique* C in the ring R is *clique*(R). A ring R is a coloring if $\chi(R)$ is finite.

For graph theoretic fundamentals readers may go through any standard book of graph theory including Chartrand [2]. Throughout the paper R will mean a ring with unity not necessarily commutative and $R_R({}_RR)$ will mean the right (left) module structure of R over itself.

An element $x(\neq 0) \in R$ *is r-freeing* (right freeing), if for any $a, b \in R$, $ab = 0$ gives $ax = 0$ and *l-freeing* (left-freeing), if $ab = 0$ gives $xb = 0$. A subset S $(\neq 0)$ of R is *r-freeing* (*l-freeing*) if its every non zero element is *r-freeing* (*l-freeing*). An element $x \in I$ (an ideal of R) with $xI \neq 0$ is *r-I freeing* if for any $a, b \in I$, $ab = 0$ gives $ax = 0$ and *l-I-freeing* if $ab = 0$ gives $xb = 0$. A subset S $(\neq 0)$ of I is *r-I-freeing* (*l-I-freeing*) if its every non zero element is *r-I-freeing* (*l-I-freeing*). An element $x \in S$ $(\subseteq R)$ is *I-freeing* if $xI \neq 0$ and for any $a, b \in I$, $ab = 0$ gives $ax = 0$ or $xb = 0$. A subset S of R is *locally-I-freeing* if for any $x(\neq 0) \in S$, x is *I-freeing* i.e. $xI \neq 0$ and for any $a, b \in I$, $ab = 0$ give $ax = 0$ or $xb = 0$. A subset S of R is *I-freeing* if $SI \neq 0$ and $ab = 0$ gives $aS = 0$ or $Sb = 0$.

An element $x \in R$ is left finite (right finite) if $Rx(xR)$ is finite. It is to be noted that in case of a non-commutative ring R for every $r \in R$, $rR = R$, then R is a division ring. On the other hand, one can say in case of a division ring R, $aR = R$, $\forall a \in R$. Moreover for some more general cases one would like to note Dheena [3], the notion of a *Duo-Near-ring* (*sub-commutative*) which may be considered as an analogous to what has been mentioned above. This justifiably motivates us as it is seen in case of a nearring [3] to give the notion of the *left* (*right*) *sub-commutativity* in a ring R as follows. A ring R is *left-sub-commutative* if for $a, b \in R$ we have $d \in R$ such that $ab = da$ (*right sub-Commutative* if for $a, b \in R$, $\exists c \in R$ such that $ab = bc$). The ring R is *sub-commutative*, if for $a, b \in R$ we have $c, d \in R$ such that $ab = bc = da$. An annihilator of $S(\subseteq R)$ in R is $Ann(S) = \{r \in R | Sr = 0\}$. A sub-module S of $R_R({}_RR)$ is prime if for every sub-module S_i of S, $AnnS_i = AnnS$. The intersection of all prime ideals of the ring R is the *nil-radical* of R. A ring R is *prime sub-commutative* in the sense that for any subset T and prime sub-module S of R_R, $TS = ST$.

A subset $T(S) = \{s \in S | sr = 0$ *for some non zero* $r \in R\}$ of S is the *torsion subset* of S. S is *torsion-free* if $T(S) = 0$ [Goodearl. Page 5, [4]]. A ring R is *reduced* if its *nil-radical* $J = 0$.

2. PRELIMINARIES

Proposition 2.1. An element $x \in R$ is *r-freeing* if and only if x is *r-R-freeing*.

Proof. Assume $x \in R$ is *r-freeing*. Then $x \neq 0$. Clearly $xR \neq 0$. Now for any $a, b \in R$, $ab = 0$ gives $ax = 0$ as x is *r-freeing*. Thus x is *r-R-freeing*. Conversely, assume that x is an *r-*

R-freeing element. Therefore $xR \neq 0$ which gives $x \neq 0$. And for any $a, b \in R$, $ab = 0$ gives $ax = 0$. Thus x is *r-freeing.*

Note:

1) An element $x \in R$, is *l-freeing* if and only if x is *l-R-freeing.*

2) Every *locally N-freeing* set is always an *N-freeing* set but the converse is not true. Converse is true only in case of a singleton set $S = \{x\}$.

3) Every *N-freeing* set contains a *locally N-freeing* set. If an *N-freeing* set contans a locally *N-freeing* set with highest cardinality n, ten it is called an *N-freeing* set with degree n.

Proposition 2.2. Let R be a ring with unity and S be a subset of R. If S is a prime sub-module of R_R, then S is *l-freeing.*

Proof. S is a prime sub-module of R. Therefore $S \neq 0$. Let $x(\neq 0) \in S$ and $a, b \in S$ such that $ab = 0$ gives $Sab = 0$. Hence $b \in r(Sa) = r(S)$. Thus $Sb = 0$ implies $xb = 0$. Thus S is *l-freeing.*

Note. 1. If S is a prime sub-module of ${}_RR$, then S is an *r-freeing* set.

Noe. 2. R is a ring with unity and S (a right ideal of R) is an *l-R-freeing set.* Then S is a prime sub-module of R_R.

Proof. Since S is *l-R-freeing,* $S \neq 0$. Let S_t be any sub-module of the module S. Then $r(S) \subseteq r(S_t)$. Conversely, let $\alpha \in r(S_t)$. Then $S_t\alpha = 0$. i.e. $S_t\alpha = 0 \forall s_t \in S_t$. Since S is *l-R-freeing,* $S\alpha = 0$. Hence $\alpha \in r(S)$. Thus $r(S_t) \subseteq r(S)$ giving thereby S is a prime sub-module or R_R.

Lemma 2.3. Let R_1 and R_2 be two prime sub-modules of R_R, where R is a *torsion-free* prime sub-commutatie ring, such that $r(R_1)$ and $l(R_2)$ are different maximal ideals. Then $R_1R_2 = 0$.

Proof. Assume $R_1R_2 \neq 0$. Then $R_2 \not\subset r(R_1)$. Therefore there exists a $r_2 \in R_2$ such that $r_2 \notin r(R_1)$. Let $x \in r(R_1) : R_2$. Then $R_2x \subseteq R_1$ which gives $r_2x \in r(R_1)$. Thus $R_1r_2x = 0$ i.e. $x \in r(R_1r_2) = r(R_1)$. Hence $x \in r(R_1)$. Conversely, let $x \in r(R_1)$. So $R_1x = 0 \Rightarrow (R_1x)R_2 = 0$. As R is prime sub-commutative $R_1R_2x = 0$ which gives $R_2x \subseteq r(R_1)$. Hence $x \in r(R_1) : R_2$ implying $r(R_1) \subseteq r(R_1) : R_2$ giving thereby $r(R_1) = r(R_1) : R_2$.

Next for any $x \in r(R_1)$, $R_1x = 0$ gives $(R_1x)R_2 = 0$. Consequently $R_1R_2x = 0$. Thus $x \in r(R_1R_2)$. Conversely let $x \in r(R_1R_2)$ gives $R_1R_2x = 0$. i.e. $R_2x \subseteq r(R_1)$. Thus $x \in r(R_1) : R_2 = r(R_1)$. Hence $r(R_1R_2) = r(R_1)$. Now $\alpha \in r(R_1R_2)$ implies $R_1R_2\alpha = 0$. i.e. $R_1(R_2\alpha) = 0$. Therefore $R_2\alpha = 0$ [R_1 is torsion-free] gives $\alpha R_2 = 0$ [for any $\alpha r_2 \in \alpha R_2$, $\alpha r_2 = r'_2\alpha = 0$

for some $r'_2 \in R_2$]. i.e. $\alpha \in l(R_2)$. Thus $r(R_1R_2) \subseteq l(R_2)$. Again $x \in l(R_2)$ gives $xR_2 = 0$. Therefore $R_1(xR_2) = 0 \Rightarrow R_1R_2x = 0$ [$r_1r_2x = r_1xr'_2 = r_10 = 0$ for some $r'_2 \in R_2$]. Hence $x \in r(R_1R_2)$. Thus $l(R_2) \subseteq r(R_1R_2)$ giving thereby $l(R_2) = r(R_1R_2)$. So $l(R_2) = r(R_1R_2) = r(R_1)$, a contradiction as $r(R_1)$ and $l(R_2)$ are different maximal ideal. Thus $R_1R_2 = 0$.

Lemma 2.4. Let R be a ring and S a finite sub-module of R_R. Then $\frac{S:x}{Annx}$ is a finite R-module. [$S : x = \{r \in R \setminus xr \in S\}$].

Proof. We know, $Ann_R(x + S) = \{r \in R | (x + S)r = S\} = \{r \in R | xr + S = S\} = \{r \in R | xr \in S\} = S : x$ Now the map $f : Ann_R(x + S) \to x(Ann_R(x + S))$ defined by $f(\alpha) = x\alpha$ is onto and $Ker(f) = \{\alpha \in Ann_R(x + S) | f(\alpha) = 0\} = \{\alpha \in Ann_R(x + S) | x\alpha = 0\} = \{\alpha \in Ann_R(x + S) | \alpha \in Ann_Rx\} = Ann_Rx[Ann_Rx \subseteq Ann_R(x + S)]$. Therefore, $\frac{Ann_R(x+S)}{Ann_Rx} \cong x(Ann_R(x+S))$, i.e. $\frac{S:x}{Ann_Rx} \cong x(S:x)$. Again, $S : x = \{r \in R | xr \in S\}$ gives $x(S : x) \subseteq S$ and since S is finite, therefore $\frac{S:x}{Ann_Rx}$ is finite.

Lemma 2.5. Let R have an infinite numbers of *right-finite* elements. Then R contains an infinite *clique* where R is a right sub-commutative ring.

Proof. Let $x_1, x_2, \ldots$ be distinct *right-finite* elements in R. Then the elements $x_1{\cdot}x_2, \ldots, x_1{\cdot}x_n$ belong to the finite ideal $x_1{\cdot}R$. Now for some infinite subsequence a_n from $\{2, 3, \ldots, n\ldots\}$, we have $x_1.x_{a_1} = x_1.x_{a_2} = \ldots$ Consider the sequence $x_{a_1}, x_{a_2}, \ldots$ and repeat the procedure. Continuing in this way we construct a subsequence $y_1, y_2, \ldots, y_n, \ldots$ of the sequence $x_1, x_2, \ldots$ such that $y_i{\cdot}y_j = y_i{\cdot}y_k$ where $k, j > i$. In this sebsequence $y_1 = x_1$, $y_2 = x_{a_1}$ etc. Put $z_{ij} = y_i - y_j$. Now $z_{ij}{\cdot}z_{kr} = z_{ij}z_{kr} = (y_i - y_j)(y_k - y_r) = y_i(y_k - y_r) - y_j(y_k - y_r) = 0$. We now construct an infinite *clique* of R. Consider $z_{1,2}.z_{3\,4} = z_{1,2}.z_{3.5} = 0$. Since, $z_{1,2} \neq z_{3,5}$, at least one of $z_{3,4}$ and $z_{3,5}$ is different from $z_{1,2}$. If for instance $z_{1,2} \neq z_{3,5}$ then $\{z_{1,2}, z_{3,5}\}$ is a *clique* with two elements. We observe that $z_{6,7}, z_{6,8}, z_{6,9}$ are different and if for $z_{6,9} \notin \{z_{1,2}, z_{3,5}\}$ then $\{z_{1,2}, z_{3,5}, z_{6,9}\}$ is a *clique* with three elements. Continuing in this way we get an infinite *clique* of R.

It is easy to see the following lemma.

Lemma 2.6. Let R be a *right sub-commutative* ring and I be a finite ideal of R. Then R contains an infinite clique if and only if $\frac{R}{I}$ contains an infinite clique.

Lemma 2.7. Let R be a *right-sub commutative* ring. If the ring R contains a nilpotent element, which is not *right-finite,* then R contains an infinite *clique.*

Proof. Let $x \in R$ be a nilpotent element. Ten $x^n = 0$ for some smallest integer $n > 0$. The proof goes by induction on n. If $x^2 = 0$ and xR is infinite, then xR itself is a *clique*. Suppose the hypothesis is true for elements of nil potency less than n. Now let $x \in R$ be such that $x^n = 0$ and x is not *right-finite.* Then xR is infinite. Put $y = x^2$. Then $y^{n-1} = 0$. If yR is infinite we conclude that R has an infnite *clique*. Assume that yR is finite. Denote $\overline{xR} = \frac{xR}{yR}$. Then $\overline{xR}$ is an infinite *clique* in $\overline{R} = \frac{R}{yR}$, then R has an infinite *clique.*

Lemma 2.8. Let R be a *right sub-comutative ring.* If R is *reduced* and is without infinite *directed-clique,* then R has the a.c.c. on right annihilators of subsets of R.

Proof. Assume an ascending chain of right annihilators of subsets of S as $r(S_1) \subset r(S_2) \subset r(S_3) \subset$...which is not stationary. Let $x_i \in r(s_i) \backslash r(S_{i-1})$, $i = 2, 3, \ldots$. Then $x_i \in r(s_i)$ and $x_i \notin r(S_{i-1})$. Consider $y_n = s_{n-1}x_n$ where $s_{n-1} \in S_{n-1}$. Then $y_i = s_{i-1}x_i$, $y_j = s_{j-1}x_j$. Now for $i > j$, $r(S_j) \subset r(S_i)$. Therefore $x_j \in r(s_{i-1})$. i.e. $S_{i-1}x_j = 0$. Then $y_iy_j = 0$. Now $z_n = y_{i-n}$ for all n. And if $y_i = y_j$, then $y_i^2 = y_j^2 = 0$ and is possible only when $y_i = 0 = y_j$ as R is with zero *nil-radical.* Now since $r(S_i) \subset r(S_2) \subset$... is not stationary therefore $\{z_n\}_{n=1}^{\alpha}$ is an infinite *directed-clique* of R, a contradiction. Thus R has the a.c.c. on right annihilators of subsets of R.

Lemma 2.9. Let R be a *torsion-free* prime sub-commutative ring. If R is *reduced* and $\chi(R) < \propto$, then the zero ideal of R is a finite intersection of prime ideals.

Proof. Assume $\chi(R) < \propto$. Now since R is *reduced,* R has the a.c.c. on right annihilators of sub-modules of R_R of the form $r(S)$. Let $r(S_i)$, $i \in I$ be the distinct maximal members of the family $\{r(S) | S \neq 0\}$ where S_i's are prime sub-modules of R. Clearly each $r(S_i)$ is a prime ideal in R. We claim I is finite. Suppose that I is not finite. Then $S_iS_j = 0 \forall i, j \in I$ which gives an infinite clique or R, a contradiction. Hence the claim. Now we consider a subset $T(\neq 0) \subseteq R$. Then $r(T) \subseteq r(S_i)$ for some $i \in I$. If $TS_i = 0$ then $S_i \subseteq r(T) \subseteq r(S_i))$ gives $S_i^2 = 0$. Therefore $S_i = 0$, a contradiction. Hence $TS_i \neq 0$. Now since R is *prime sub-commutative,* $S_iT \neq 0$, we have $T \not\subset r(S_i) \forall i \in I$. Thus $\bigcap_{i \in I} r(S_i) = 0$. Hence the theorem.

Lemma 2.10. Let R be a coloring which is *right sub-commutative.* If *nil-radical* of R is *torsion-free* then it is finite.

Proof. Since $\chi(R) < \propto$, R has no infinite *clique*. Suppose *nil-radical* J is not finite. Then JR is not a finite sub-module [as J is *torsion free* subset of R]. Now if every element of JR is *right-finite,* then JR contains an infinite *clique* [Lemma 2.5]. Hence J contains an infinite *clique,* since $JR \subseteq J$. If the sub-module JR contains an element which is not *right-finite,* then JR contans an infinite *clique* [Lemma 2.7] implying the existence of an infinite *clique* in J too, a contradiction giving thereby the *nil-radical* J of R is finite.

3. MAIN RESULTS

Theorem 3.1. Let R be *torsion-free* and right as well as prime *sub-commutative.* Then R has the a.c.c. on right annihilators of subsets of R if $\chi(R) < \infty$.

Proof. Consider an ascending chain of right annihilators of subsets of S say $r(S_i) \subset r(S_2) \subset ...$ Now the *nil-radical* J or R is finite [lemma 2.10]. Let $S_i \not\subset J$ for $i = 1, 2, ...$ Now J is a finite intersection of prime ideals [lemma 2.9]. Let $J = P_1 \cap P_2 \cap .... \cap P_n$ where $P'_i s$ are prime ideals of R. Then for any subset $S \subseteq R$ we get $J : S = (P_1 : S) \cap (P_2 : S) \cap ... \cap (P_n : S)$. We claim that the family $\{J : S | S \subseteq R\}$ is finite. Now for $S \subseteq R$ either $S \subseteq J$ or $S \not\subset J$. If $S \subseteq J . J : S = R$.

[for $r \in R$, $Sr \subseteq J \Rightarrow r \in J : S \Rightarrow R \subseteq J : S \Rightarrow R = J : S$]. If $s \not\subset J$ then $J : S$ is nothing but one of the elements of the family $\{P_1 : S, P_2 : S, ..., P_n : S, (P_1 : S) \cap (P_2 : S), ... (P_i : S) \cap (P_j : S), ...(P_1 : S) \cap \}(P_2 : S) \cap ... \cap (P_n : S)\}$. Thus the family $\{J : S | S \subseteq R)$ is a finite family. Consequently there exists a subsequence $\{T_i\}$ of $\{S_i\}$ for which $J : T_1 = J : T_2 = ...$ Consider $r(T_1) \subset r(T_2) \subset r(T_3) \subset ...$ and $J : T_1 = J : T_2 = ...$ This implies that $r(T_1) \subset r(T_2) \subset r(T_3) \subset ... \subset J : T_1 = J : T_2 = ... [x \in r(T_i) \Rightarrow T_i x = 0 \Rightarrow x \in J : T_i]$ contradicting the fact that $\frac{J:T_1}{r(T_1)}$ is finite [lemma 2.4].

Theorem 3.2. Let R be a coloring as in the theorem 3.1 and S, a non zero sub-module of the module R_R. Then S contains an *l-freeing* set.

Proof. R be a right sub commutative *torsion-free* ring which is a coloring. Therefore R has the a.c.c. on right annihilators of subsets of R. Now S is a non zero sub module of the module R_R. We consider the ascending chain of annihilators of sub-module S such as $r(S_i) \subseteq r(S_2) \subseteq ...$ which is stationary. Let $r(T)$ be the maximal element. Thus $r(T)$ is a prime ideal. We claim T is a prime sub-module of S. Let T_1 be any sub-module of T. Then $r(T) \subseteq r(T_1)$ which gives $r(T) = r(T_1)$ [$r(T)$ is maximal]. Thus T is a prime sub-module of S. Hence T is an *l-freeing set* [Proposition 2.2].

Now we give some insight to a restricted case:

Let N be an ideal of the ring R. Consider the set $\{S_i(\neq 0) \subseteq N | S_i$, *a right ideal of* $R\}$. Then we write $J_N = \cap \max r(S_i)$.

Theorem 3.3. Let N be an ideal of right sub commutative ring R which is a coloring such that N is not contained in the J_N of R. Then N contains an *l-N-freeing* set if R is *torsion-free.*

Proof. Here N is an ideal of R such that N is not contained in the J_N of R. i.e. $N \not\subset J_N = \cap \max r(S_i)$. Therefore there exists a S_k, a right ideal of R such that $N \not\subset r(S_k)$. We claim S_k is an *l-N-freeing* set. Let $x(\neq 0) \in S_k$, we have $N \not\subset r(x)$ [since $r(S_k) = r(x)$] which gives $xN \neq 0$. Let $a, b \in S_k$ such that $ab = 0$. Therefore $S_kab = 0$ gives $b \in r(S_ka) = r(S_k) = r(x)$. Thus $xb = 0$. Hence x is an *l-N-freeing* element. Thus every non zero element of S_k is *l-N-freeing.* Hence S_k is an *l-N-freeing* set.

Theorem 3.4. Let $I = xR$ be a non zero ideal in a right sub commutative coloring R whcih is reduced. If $I^2 \neq 0$, then I contains an *l-I-freeing* set.

Proof. Let $I = xR$ be an ideal in right sub commutative coloring R which is reduced. Here $x^2 \neq 0$. Consider the family $\zeta = \{S_ix^2(\neq 0) | S_i, \textit{a right ideal of } R\}$. $\zeta \neq \phi$ as $Rx^2 \in \zeta$. As R has the a.c.c. on right annihilators of subsets of R, let $r(Tx^2)$ be a maximal ideal for some right ideal T of R. Now we claim Tx is *l-I-freeing* set. Let $a, b \in I$ be such that $a = xr$ and $b = xs$ for some $r, s \in R$. Now $ab = 0$ gives $xr.xs = 0$. Therefore $Tx^2r's = 0$ [for some $r' \in R$.]. Thus $s \in r(Tx^2r') = r(Tx^2)$ gives $Tx^2s = 0$ i.e. $(Tx)(xs) = 0$. Thus $Tx.b = 0$. Again $Tx^2 \neq 0$. Therefore $(Tx)I \neq 0$. Thus Tx an *l-I-freeing* set.

Theorem 3.5. Let I be an ideal in a right sub commutative ring R and S (right ideal of R) $\subseteq I$ is *l-I-freeing.* Put $I' = r(S) \cap I$

1) If $S^2 = 0$, then *clique* (I') = *clique*(I).

2) If $S^2 = 0$, then $\chi(I') = \chi(I)$.

Proof.

1) Assume that $S^2 = 0$. Then $S \subseteq r(S)$ gives that $S \subseteq r(S) \cap I = I'$. Let *clique* $(I) = n$ and choose a maximal *directed-clique* $C = \{y_1, y_2, \ldots, y_n\}$ in I. Let $x \in S$ then $x^2 = 0$. If $x \in C$, say $x = y_j$ for some j, $1 \leq j \leq n$ then $\{y_1, y_2, \ldots, y_{j-1}, x, y_{j+1}, \ldots, y_n\}$ is a *directed-clique* in I'. Since $xy_k = 0, \forall 2 \leq i \leq n$ as S is *l-freeing* we have $y_k \in r(x) = r(S)$. Therefore $y_k \in r(S) \cap I = I'$. Again $x^2 = 0$. Thus $x^2y_1 = 0$ which gives $xy_1 \in r(x) = r(S)$. Hence $Sxy_1 = 0$. Thus $y_1 \in r(Sx) = r(S)$ which gives $y_1 \in I'$. Thus *clique* $(I') = n$. Thus *clique* (I') = *clique* (I).

If $x \notin C$ then $xC \neq 0$, since C is maximal *directed-clique* in I. For otherwise $\{x, y_1, y_2, \ldots, y_n\}$ will be a *directed-clique* of I. Now $xC \neq 0$. Let $xy_j \neq 0$ for some j. Consider the set $C' = \{x, y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n\}$. Then C' is a *directed-clique* in I'. Thus *clique* (I') = *clique*(I).

2) Assume $S^2 = 0$ and consider $\chi(I') = \chi(I)$. Color first the elements of I'. If $y \in I \setminus I'$ assign to y the same color as to x. This shows that $\chi(I') = \chi(I)$. [As in case of Beck [1]].

Theorem 3.6. Let I be an ideal in a right sub commutative ring R and S (right ideal of R) $\subseteq I$ is *l-I-freeing*. Put $I' = r(S) \cap I$

1) If $S^2 \neq 0$, then *clique* $(I') =$ *clique* $(I) - 1$.

2) If $S^2 \neq 0$, then $\chi(1') = \chi(I) - 1$.

Proof.

1) $S^2 \neq 0$ gives $S \not\subset I'$ [since $S \not\subset r(S)$]. Let $x \in S$ such that $x \notin I'$ i.e. x is an *l-freeing* element such that $x \notin I'$. Then *clique*$(I') \leq$ *clique*$(I) - 1$, since to any *directed-clique* in I', we may adjoin x and create a larger *directed-clique* in I.

Now suppose *clique*$(I') \leq$ *clique*$(I) - 1$. Let *clique*$(I) = n$ and $C = \{y_1, y_2, ..., y_n\}$ be a maximal *directed-clique* in I. Now $y_iy_i = 0$ for $i \leq j$. Since x is *l-freeing* therefore, $xy_i = 0$, $\forall 2 \leq i \leq n$ which gives $\{x, y_2,...,y_n\}$ is an another maximal *directed-clique* in I. Now $y_i \in r(x) = r(S)$, $\forall 2 \leq i \leq n$.

Hence $y_i \in$ I′. Thus $\{y_2,...,y_n\}$ is a maximal *directed-clique* in I'. Thus we get a maximal *directed-clique* of I' having $n - 1$ numbers of elements. Thus *clique*$(I') =$ *clique*$(I) - 1$.

2) Assume $S^2 \neq 0$. Let $x \in S$ such that $x \notin I'$. Now color first the elements of I'. Clearly $x \in I \backslash I'$ and we need one color more than $\chi(I') = n$ colors to color x because for any maximal *directed-clique* $\{y_1, y_2, ...,y_n\}$ of I'. Here $xy_i = 0$ for any i, $1 \leq i \leq n$ since $y_i \in r(S) = r(x)$. Thus x is adjacent to y_i for any i. Thus $\chi(I') + 1 \leq \chi(I)$. Assume $\chi(I) - 1$. Color first the elements of I. Then we need $\chi(I) = n$ colors to color I. Let $\{y_1, y_2,....,y_n\}$ is a maximal *directed-clique* of I. Then $\{y_2,...,y_n\}$ is a maximal *directed-clique* of I', a contradiction. Thus $\chi(I') = n - 1$. Hence $\chi(I') = \chi(I) - 1$.

REFERENCES

1. Beck, Istvan, : Coloring of commutative Rings, Journal of Algebra 116, 208-226 (1988).
2. Chartrand, G at-el : Introduction to graph theory. Tata McGraw Hill.
3. Dheena, P : On Strongly regular near-rings, J. Indian Math. Soc. 49(1887), 201-208.
4. Goodearl, K. R. : Ring Theory (Nonsingular Rings and modules.), MARCEL DEKKER, INC, New York And Basel. 1976.

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FUZZY TRANSLATIONAL SUBSET OF A Γ-RING

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ABSTRACT : In this paper we define ideals of a Γ-ring M generated by a fuzzy subset together with an element of the Γ-ring M and an element of Γ and study their properties. Also the notions of units, associates, prime element, irreducible element are generalized with respect to a fuzzy subset.

Key words and pharases : Fuzzy subset, Γ-ring, Translational invariant fuzzy subset.
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1. PRELIMINARIES

The notion of fuzzy ideal in Γ-ring was introduced by Y.B. Jun and C.Y.Lee [6]. In this paper we introduce the notion of fuzzy translational subset of a Γ-ring M and generalize some notions of a ring to a Γ-ring.

2. SOME BASIC DEFINITIONS AND EXAMPLES

Definition 2.1. [1] Let M and Γ be two additive abelian groups. M is called a Γ-ring if there exists a mapping $f: M \times \Gamma \times M \to M$, $f(a, \alpha, b)$ is denoted by $a\alpha b$, $a, b \in M, \alpha \in \Gamma$, satisfying the following conditions for all $a, b, c \in M$ and for all $\alpha, \beta, \gamma \in \Gamma$; $(a + b)\alpha c = a\alpha c + b\alpha c$, $a(\alpha + \beta)b = a\alpha b + a\beta b$, $a\alpha(b + c) = a\alpha b + a\alpha c$, and $a\alpha(b\beta c) = (a\alpha b)\beta c$.

Definition 2.2. [1] A subset A of a Γ-ring M is called a left (resp. right) ideal of M if A is an additive subgroup of M and $m\alpha a \in A$ (resp. $a\alpha m \in A$) for all $m \in M$, $\alpha \in \Gamma$, $a \in A$. If A is a left and a right ideal of M then A is called a two sided ideal of M or simply an ideal of M.

Definition 2.3. A Γ-ring M is said to be commutative if $a\gamma b = b\gamma a$ for all, $a, b \in M, \gamma \in \Gamma$.

Definition 2.4. [2] Let M be a Γ ring and F be the free abelian group generated by $\Gamma \times M$. Then $A = \{\sum_i n_i(\gamma_i, x_i) \in F : a \in M \Rightarrow \sum_i n_i a\gamma_i x_i = 0\}$ is a subgroup of F. Let $R = F/A$ be the factor group of F by A. Let us denote the coset $(\gamma, x) + A$ by $[\gamma, x]$. It can be verified that $[\alpha, x] + [\beta, x] = [\alpha + \beta, x]$ and $[\alpha, x] + [\alpha, y] = [\alpha, x + y]$, for all $\alpha, \beta \in \Gamma$ and x, y

$\in M$. We define a multiplication in R by $\sum_i [\alpha_i, x_i] \sum_j [\beta_j, y_j] = \sum_{i,j} [\alpha_i, x_i\beta_j y_j]$. Then R forms a ring. This ring R is called the right operator ring of the Γ ring M. Similarly we can construct left operator ring L of M, where $L = \{\sum_i [x_i, \alpha_i] | x_i \in M, \alpha_i \in \Gamma\}$ and the multiplication on L is defined by $\sum_i [x_i, \alpha_i] \sum_j [y_j, \beta_j] = \sum_{i,j} [x_i\alpha_i y_j, \beta_j]$. For the subsets $N \subseteq M$, $\Phi \subseteq \Gamma$, we denote by $[\Phi, N]$ the set of all finite sums $\Sigma_i[\gamma_i, x_i]$ in R where $\gamma_i \in \Phi$ and $x_i \in N$ and we denote by $[(\Phi, N)]$ the set of all elements $[\phi, x]$ in R, where $\phi \in \Phi$ and $x \in N$. Thus in particular, $R = [\Gamma, M]$ and $L = [M, \Gamma]$. If there exists an element $\sum_i [\delta_i, e_i] \in R$ such that $\sum_i x\delta_i e_i = x$ for every element x of M then it is called the right unity of M. It ca be verified that $\sum_i [\delta_i, e_i]$ is the unity of R. Similarly we can define the left unity $\sum_j [f_j, \gamma_j]$ which is the unity of the left operator ring L.

Definition 2.5. [6] A nonempty fuzzy subset μ (i.e., $\mu(x) \neq 0$ for some $x \in M$) of a Γ ring M is called a fuzzy left (resp.right) ideal of M if i) $\mu(x - y) \geq \min\{\mu(x), \mu(y)\}$, ii) $\mu(x\alpha y) \geq \mu(y)(resp. \mu(x\alpha y) \geq \mu(x))$ for $x, y \in M$ and for all $\alpha \in \Gamma$.

A non-empty fuzzy subset μ of a Γ-ring M is called a fuzzy ideal of M if it is a fuzzy left ideal and a fuzzy right ideal of M.

Definition 2.6.[3] Let μ, σ be two fuzzy subsets of M. Then the sum $\mu \oplus \sigma$ and composition $\mu_0\sigma$ of μ and σ are defined as follows:

$$(\mu \oplus \sigma)(x) = \sup_{x=u+v} [\min[\mu(u), \sigma(v)]] \text{ for } u, v \in M$$

$$= 0 \text{ otherwise.}$$

$$(\mu_0\sigma)(x) = \sup[\min[\min[\mu(u_i), \sigma(v_i)]]], \ 1 \leq i \leq n, \ x = \sum_{i=1}^{n} u_i\gamma_i v_i, u_iv_i \in M \text{ and } \gamma_i \in \gamma$$

$$= 0 \text{ otherwise.}$$

Definition 2.7. [7] Let μ, σ be two fuzzy subsets of M. Then the product $\mu\Gamma\sigma$ of μ and σ is defined by

$$(\mu\Gamma\sigma)(x) = \sup_{x=u\gamma v}[\min[\mu(u), \sigma(v)], \text{ for } u, v \in M \text{ and } \gamma \in \Gamma$$

$$= 0 \text{ otherwise.}$$

We denote the set of all fuzzy ideals of M and the set of all fuzzy ideals of R by $FI(M)$ and $FI(R)$ respectively.

Definition 2.8. [6] A function $f : M \to N$, where M, N are Γ-rings is said to be a Γ-homomorphism if $f(a + b) = f(a) + f(b)$, $f(a\alpha b) = f(a)\alpha f(b)$, for all $a, b \in M$, $\alpha \in \Gamma$.

Definition 2.9. [3] A commutative Γ-ring M is called a Γ-field if for every nonzero element a of M and for every nonzero elements $\gamma_1, \gamma_2 \in \Gamma$, there exists a nonzero element a' of M such that $a\gamma_1 a'\gamma_2 b = b$ for all $b \in M$.

3. FUZZY TRANSLATIONAL SUBSET OF A Γ-RING

Throughout this paper M is a Γ-ring with unity.

Definition 3.1. Let μ be a fuzzy subset of M. μ is called fuzzy left translational invariant with respet to internal addition if $\mu(x) = \mu(y)$ implies $\mu(m + x) = \mu(m + y)$, for all $x, y, m \in M$. Again μ is called fuzzy left translational invariant with respect to external multiplication if $\mu(x) = \mu(y)$ implies $\mu(m\gamma x) = \mu(m\gamma y)$, for all $x, y, m \in M$ and for all $\gamma \in \Gamma$.

Example 3.2 Consider the Γ-ring M, where $M = Z$, the ring of integers and $\Gamma = 2Z$, the ring of even integers and $x\gamma y$ denotes the usual product of the integers x, γ, y. Let μ be a fuzzy subset of M defined by $\mu(x) = 1$, if x is an even integer and $\mu(x) = .5$, if x is an odd integer. It is very easy to check that if $\mu(x_1) = \mu(x_2)$, where $x_1, x_2 \in M$ then $\mu(x_1 + y) = \mu(x_2 + y)$, $\mu(x_1\gamma y) = \mu(x_2\gamma y)$, for any $y \in M$ and for any $\gamma \in \Gamma$. So μ is a translational invariant fuzzy subset of M with respect to internal addition and external multiplication defined on M.

Example 3.3. Consider the Γ-ring M, where $M = \{[a_{ij}] : a_{ij} \in Z_2, i = 1, j = 1, 2\}$, the set of 1×2 matrices whose entries are from Z_2 and $\Gamma = \{[a_{ij}] : a_{ij} \in Z_2, i = 1, 2, j = 1\}$, the set of 2×1 matrices whose entries are from Z_2. Let μ be a fuzzy subset of M defined by $\mu([a_{ij}]) = .7$, if $a_{11} = 0 = a_{12}$ or $a_{11} = 1$, $a_{12} = 0$ and $\mu([a_{ij}]) = .3$, if $a_{11} = 0$, $a_{12} = 1$ or $a_{11} = 1 = a_{12}$. It is very easy to check that if $\mu(x_1) = \mu(x_2)$, where $x_1, x_2 \in M$ then $\mu(x_1 + y) = \mu(x_2 + y)$, $\mu(x_1\gamma y) = \mu(x_2\gamma y)$, for any $y \in M$ and for any $\gamma \in \Gamma$. So μ is a translational invariant fuzzy subset of M with respect to internal addition and external multiplication defined on M.

Definition 3.4. Let μ be a fuzzy subset of M. μ is called fuzzy right translational invariant with respect to internal addition if $\mu(x) = \mu(y)$ implies $\mu(x + m) = \mu(y + m)$, for all $x, y\ m$

$\in M$. Again μ is called fuzzy right translational invariant with respect to external multiplication if $\mu(x) = \mu(y)$ implies $\mu(x\gamma m) = \mu(y\gamma m)$, for all $x, y, m \in M$ and for all $\gamma \in \Gamma$.

Definition 3.5. Let μ be a fuzzy subset of M. μ is called fuzzy translational invariant with respect to internal addition if μ is both fuzzy left and fuzzy right translational invariant with respect to internal addition. Again μ is called fuzzy translational invariant with respect to external multiplication μ is both fuzzy left and fuzzy right translational invariant with respect external multiplication.

Remark 3.6. A fuzzy left translational invariant with respect to internal addition in a Γ-ring is also a fuzzy right translational invariant with respect to intrnal addition and vice versa.

If μ is commutative, that is $\mu(x\gamma y) = \mu(y\gamma x)$, for all $x, y \in M$, $\gamma \in \Gamma$ then μ is fuzzy left translational invariant with respect to external multiplication if and only if μ is fuzzy right translational invariant with respect to external multiplication.

4. IDEALS OF A Γ-RING GENERATED BY ELEMENT AND FUZZY SUBSET

Throught this section μ is a fuzzy subset of a Γ-ring M satisfying $\mu(x) = \mu(-x)$ for all $x \in M$.

Proposition 4.1. Let μ be a fuzzy left translational invariant with respect to both internal addition and external multiplication defined on M. Then for any $m \in M$ and for any $\gamma \in \Gamma$ the set

$$L(m, \gamma, \mu) = \{x : x \in M \text{ and } \mu(x) = \mu(y\gamma m), \text{ for some } y \in M\}$$

is a left ideal of M.

Proof. Clearly $L(m, \gamma, \mu)$ is non-empty, since $0_M \in L(m, \gamma, \mu)$ as $\mu(0_M) = \mu(0_M\gamma m)$. Let x_1, $x_2 \in L(m, \gamma, \mu)$. Then $\mu(x_1) = \mu(y_1\gamma m)$ for some $y_1 \in M$ and $\mu(x_2) = \mu(y_2\gamma m)$ for some $y_2 \in M$.

Now $\mu(x_1) = \mu(y_1\gamma m)$ implies that $\mu(x_1 - x_2) = \mu(y_1\gamma m - x_2) = \mu(x_2 - y_1\gamma m)$. Again $\mu(x_2) = \mu(y_2\gamma m) = \mu(x_2 - y_1\gamma m)$ implies that $\mu(x_2 - x_1) = \mu(y_2\gamma m - x_1) = \mu(x_1 - y_2\gamma m)$. Thus we have $\mu(x_1 - y_2\gamma m) = \mu(x_2 - y_1\gamma m)$ which implies that $\mu(x_1 - x_2) = \mu(y_1\gamma m - y_2\gamma m) = \mu((y_1 - y_2)\gamma m)$. So $x_1 - x_2 \in L(m, \gamma, \mu)$.

Also for any $y_3 \in M$ and $\gamma_1 \in \Gamma$, $\mu(y_3\gamma_1 x_1) = \mu(y_3\gamma_1(y_1\gamma m)) = \mu((y_3\gamma_1 y_1)\gamma m)$, which implies that $y_3\gamma_1 x_1 \in L(m, \gamma, \mu)$, for any $y_3 \in M$ and for any $\gamma_1 \in \Gamma$. Hence $L(m, \gamma, \mu)$ is a left ideal of M.

Similarly we can prove the following Proposition.

Proposition 4.2. Let μ be a fuzzy right translational invariant with respect to both internal addition and external multiplication defined on M. Then for any $m \in M$, $\gamma \in \Gamma$ the set

$$R(m, \gamma, \mu) = \{x : x \in M \text{ and } \mu(x) = \mu(m\gamma y), \text{ for some } y \in M$$

is a right ideal of M.

Remark 4.3. If M is a commutative Γ-ring then $L(m, \gamma, \mu) = R(m, \gamma, \mu)$ for all $m \in M$ and for all $\gamma \in \Gamma$.

Remark 4.4. We observe that for any $m \in M$ and $\gamma \in \Gamma$, the ideal $M\gamma m = \{x\gamma m : x \in M\}$ of M is contained in the left ideal $L(m, \gamma, \mu)$. Also for any $m \in M$ and $\gamma \in \Gamma$, the ideal $m\gamma M = \{m\gamma x : x \in M\}$ of M is contained in the right ideal $R(m, \gamma, \mu)$.

Definition 4.5. $L(m, \gamma, \mu)$ is called the left μ-principal ideal of M generated by m, γ and μ.

Definition 4.6. If $L(m, \gamma, \mu) = R(m, \gamma, \mu)$ for $m \in M$ and $\gamma \in \Gamma$ and $\mu \in FS(M)$, the set of all fuzzy subsets of M then the ideal is denoted by $I(m, \gamma, \mu)$ and is called the μ-principal ideal of M generated by m, γ and μ.

Definition 4.7. A Γ-ring M is caled a μ-principal ideal Γ-ring if μ is commutative and every ideal of M is a μ-principal ideal generated by some $m \in M$, $\gamma \in \Gamma$ and $\mu \in FS(M)$.

Definition 4.8. An element $a \in M$ with $\mu(a) \neq \mu(0_M)$ is called a μ_γ-unit of M, where $\gamma \in \Gamma$ if there exists an element $a' \in M$ such that $\mu(a') \neq \mu(0_M)$ and $\mu(a\gamma a'\gamma m) = \mu(m) = \mu(a'\gamma a\gamma m)$ for all $m \in M$.

From the definition it follows that $\gamma \neq 0_\Gamma$.

In a Γ-field every element $a(\neq 0_M)$ is a μ_γ-unit for all $\gamma(\neq 0_\Gamma) \in \Gamma$.

Proposition 4.9. If a is a μ_γ-unit of M, then $L(a, \gamma, \mu) = R(a, \gamma, \mu) = M$, for $\gamma \in \Gamma$.

Proof. As a is a μ_γ-unit of M, there exists an element $a' \in M$ such that $\mu(a') \neq \mu(0_M)$ and $\mu(a\gamma a'\gamma m) = \mu(m) = \mu(a'\gamma a\gamma m)$ for all $m \in M$. Let $x \in M$. Then $\mu(x) = \mu(a\gamma a'\gamma x)$, which implies that $x \in R(a, \gamma, \mu)$, since $a'\gamma x \in M$. Therefore $M \subseteq R(a, \gamma, \mu)$. Similarly $M \subseteq L(a, \gamma, \mu)$. Hence $L(a, \gamma, \mu) = R(a, \gamma, \mu) = M$, for $\gamma \in \Gamma, \gamma \neq 0_\Gamma$.

Proposition 4.10. Let μ be a fuzzy translational invariant with respect to external multiplication defined on M and $a, b \in M$. Then $a \in L(b, \gamma, \mu)$ for some $\gamma \in \Gamma$ implies $L(a, \gamma, \mu) \subseteq L(b, \gamma, \mu)$ and $a \in R(b, \gamma, \mu)$ for some $\gamma \in \Gamma$ implies that $R(a, \gamma, \mu) \subseteq R(b, \gamma, \mu)$.

Proof. Let $a \in L(b, \gamma, \mu)$, then $\mu(a) = \mu(x\gamma b)$, for some $x \in M$. Let $m \in L(a, \gamma, \mu)$. Then $\mu(m) = \mu(y\gamma a)$, for some $y \in M$. Now $\mu(a) = \mu(x\gamma b)$ implies that $\mu(y\gamma a) = \mu(y\gamma x\gamma b)$. Thus $\mu(m) = \mu(y\gamma x\gamma b)$, which implies that $m \in L(b, \gamma, \mu)$. Hence $L(a, \gamma, \mu) \subseteq L(b, \gamma, \mu)$.

Similarly we can prove $R(a, \gamma, \mu) \subseteq R(b, \gamma, \mu)$.

Remark 4.11. We observe that $L(0_M, \gamma, \mu) = \{m \in M : \mu(m) = \mu(0_M)\} = \mu_0$, for any $\gamma \in \Gamma$.

Proposition 4.12. Let μ be a fuzzy translational invariant with respect to external multiplication defined on M and $a, b \in M$. Then $\mu(a) = \mu(b)$ implies that $L(a, \gamma, \mu) = L(b, \gamma, \mu)$ and $R(a, \gamma, \mu) = R(b, \gamma, \mu)$, for and $\gamma \in \Gamma$.

Proof. Let $\mu(a) = \mu(b)$. Suppose $m \in L(a, \gamma, \mu)$. Then $\mu(m) = \mu(x\gamma a)$, for some $x \in M$. Now $\mu(a) = \mu(b)$ implies $\mu(x\gamma a) = \mu(x\gamma b)$. Hence $\mu(m) = \mu(x\gamma b)$, so $m \in L(b, \gamma, \mu)$. Thus $L(a, \gamma, \mu) \subseteq L(b, \gamma, \mu)$. Similarly we can show that $L(b, \gamma, \mu) \subseteq L(a, \gamma, \mu)$. Consequently, $L(a, \gamma, \mu) = L(b, \gamma, \mu)$.

Similarly, we can prove $R(a, \gamma, \mu) = R(b, \gamma, \mu)$.

In the next two sections M is assumed to be a commutative Γ-ring with right and left unities and μ be a fuzzy translational invariant with respect to both internal addition and external multiplication defined on M satisfying $\mu(x) = \mu(-x)$ for all $x \in M$. Henceforth, the ideal generated by $a \in M$, $\gamma \in \Gamma$ with respect to μ is denoted by $I(a, \gamma, \mu)$ and it is assumed that $a \in I(a, \gamma, \mu)$ for all $\gamma \in \Gamma$.

5. μ-DIVISORS OF ZERO, μ-ASSOCIATES

Definition 5.1. An element $a \in M$ with $\mu(a) \neq \mu(0_M)$ is said to be μ_γ-divisors for zero for $\gamma \in \Gamma$, $\gamma \neq 0_\Gamma$ if there exists some $b \in M$ with $\mu(b) \neq \mu(0_M)$ such that $\mu(a\gamma b) = \mu(0_M)$.

Henceforth we shall assume that M contains no μ_γ-divisors of zero.

Definition 5.2. Let $a, b \in M$ and $\mu(a) \neq \mu(0_M)$. We say that a divides b with respect to μ and $\gamma \in \Gamma$ or a is a μ_γ-divisors of b, written as $(a / b)_{\mu_\gamma}$, if either $\mu(a) = \mu(b)$ or there exists $c \in M$ such that $\mu(b) = \mu(a\gamma c)$.

Theorem 5.3. Let $a, b \in M$ be such that $\mu(a) \neq \mu(b)$ and $\mu(a) \neq \mu(0_M)$. Then $(a / b)_{\mu_\gamma}$ if and only if $I(b, \gamma, \mu) \subseteq I(a, \gamma, \mu)$, for $\gamma \in \Gamma$.

Proof. Suppose that $(a / b)_{\mu_\gamma}$. Then $\mu(b) = \mu(c\gamma a)$ for some $c \in M$, which implies that $b \in I(a, \gamma, \mu)$ and therefore $I(b, \gamma, \mu) \subseteq I(a, \gamma, \mu)$.

Conversely, let $I(b, \gamma, \mu) \subseteq I(a, \gamma, \mu)$. As $b \in I(b, \gamma, \mu) \subseteq I(a, \gamma, \mu)$ hence $\mu(b) = \mu(c\gamma a)$ for some $c \in M$. Also $\mu(a) \neq \mu(0_M)$. Hence $(a / b)_{\mu_\gamma}$.

Definition 5.4. Let $a, b \in M \setminus \mu_0$ be such that $\mu(a) \neq \mu(b)$. We say that a and b are μ-associates with respect $\gamma \in \Gamma$ if $(a/b)_{\mu_\gamma}$ and $(b/a)_{\mu_\gamma}$.

Proposition 5.5. Let $a, b \in M \setminus \mu_0$. Then a, b are μ-associates with respect to $\gamma \in \Gamma$ if and only if $\mu(a) = \mu(b\gamma u)$, for some μ_γ-unit $u \in M$.

Proof. Let a, b be μ-associates with respect to γ. Then $(a/b)_{\mu_\gamma}$ and $(b/a)_{\mu_\gamma}$. So $\mu(b) = \mu(a\gamma d)$ and $\mu(a) = \mu(b\gamma c)$ for some $c, d \in M$. Hence,

$$\mu(a) = \mu(b\gamma c) = \mu(a\gamma d\gamma c)$$
$$\Rightarrow \mu(a\gamma x) = \mu(a\gamma d\gamma c\gamma x), \text{ for all } x \in M$$
$$\Rightarrow \mu(a\gamma x - a\gamma d\gamma c\gamma x) = \mu(0_M)$$
$$\Rightarrow \mu(a\gamma(x - d\gamma c\gamma x)) = \mu(0_M)$$
$$\Rightarrow \mu(x - d\gamma c\gamma x) = \mu(0_M), \text{ since } \mu(a) \neq \mu(0_M) \text{ and } M \text{ is without } \mu_\gamma\text{-zero divisor}$$
$$\Rightarrow \mu(x) = \mu(d\gamma c\gamma x), \text{ for all } x \in M$$
$$\Rightarrow c \text{ and } d \text{ are } \mu_\gamma\text{-units}$$

Hence $\mu(a) = \mu(b\gamma c)$, where c is a μ_γ-units in M.

Conversely, suppose that $\mu(a) = \mu(b\gamma u)$, for some μ_γ-units u in M. Now, $\mu(a) = \mu(b\gamma u)$ implies $(b/a)_{\mu_\gamma}$. Since u is a μ_γ-unit, there exists $v \in M \setminus \mu_0$ such that $\mu(u\gamma v\gamma x) = \mu(x)$ for all $x \in M$. Hence $\mu(a) = \mu(b\gamma u)$ implies $\mu(a\gamma v) = \mu(b\gamma u\gamma v) = \mu(b)$. This shows that $(a/b)_{\mu_\gamma}$.

Thus $(a/b)_{\mu_\gamma}$ and $(b/a)_{\mu_\gamma}$. Hence a, b are μ_γ-associates.

Corollary 5.6. Let $a, b \in M \setminus \mu_0$. If a, b are μ_γ-associats then $I(a, \gamma, \mu) = I(b, \gamma, \mu)$.

Proof. Suppose a, b are μ_γ-associates. Then by Proposition 5.5, $\mu(a) = \mu(u\gamma b)$ for some μ_γ-unit $u \in M$. Then $a \in I(b, \gamma, \mu)$ and so $I(a, \gamma, \mu) \subseteq I(b, \gamma, \mu)$. Since u is a μ_γ-unit, there exists $v \in M \setminus \mu_0$ such that $\mu(u\gamma v\gamma x) = \mu(x)$ for all $x \in M$. Hence $\mu(b\gamma u\gamma v) = \mu(b)$. Thus $\mu(b) = \mu(a\gamma v)$, so $b \in I(a, \gamma, \mu)$. Therefore $I(b, \gamma, \mu) \subseteq I(a, \gamma, \mu)$. Consequently, $I(b, \gamma, \mu) = I(a, \gamma, \mu)$.

Definition 5.7. Suppose $a \in M \setminus \mu_0$ and a is not a μ_γ-unit for $\gamma \in \Gamma$. Then a is said to be μ_γ-irreducible if $\mu(a) \neq \mu(b)$, $\mu(a) \neq \mu(c)$ and $\mu(a) = \mu(b\gamma c)$ implies either b or c is a μ_γ-unit, where $b, c \in M$.

Definition 5.8. Suppose $a \in M \setminus \mu_0$ and a is not a μ_γ-unit for $\gamma \in \Gamma$. Then a is said to be

μ_γ-prime if $\mu(a) \neq \mu(b)$, $\mu(a) \neq \mu(c)$ and $(a / b\gamma_1 c)_{\mu_\gamma}$ implies either $(a / b)_{\mu_\gamma}$ or $(a / c)_{\mu_\gamma}$, where $\gamma_1 \in \Gamma$.

Proposition 5.9. In the Γ-ring M with no μ_γ divisors of zero, any μ_γ-prime is μ_γ-irreducible.

Proof. Let a be μ_γ-prime. Suppose $\mu(a) \neq \mu(b)$, $\mu(a) \neq \mu(c)$ and $\mu(a) = \mu(b\gamma c)$, for $\gamma \in \Gamma$, $b, c \in M$. Hence we can say that $(a / b\gamma c)_{\mu_\gamma}$. Since a is μ_γ-prime, either $(a / b)_{\mu_\gamma}$ or $(a / c)_{\mu_\gamma}$.

Let $(a / b)_{\mu_\gamma}$. As $\mu(a) \neq \mu(b)$, $\mu(b) = \mu(a\gamma d)$, for some $d \in M$. Now

$$\begin{aligned}
&\mu(a) = \mu(b\gamma c) = \mu(a\gamma d\gamma c)\\
&\Rightarrow \mu(a\gamma x) = \mu(a\gamma d\gamma c\gamma x), \text{ for } x \in M\\
&\Rightarrow \mu(a\gamma x - a\gamma d\gamma c\gamma x) = \mu(0_M)\\
&\Rightarrow \mu(a\gamma(x - d\gamma c\gamma x)) = \mu(0_M)\\
&\Rightarrow \mu(x - d\gamma c\gamma x) = \mu(0_M), \text{ since } \mu(a) \neq \mu(0_M) \text{ and } M \text{ is without } \mu_\gamma\text{-zero divisor}\\
&\Rightarrow \mu(x) = \mu(d\gamma c\gamma x) \text{ for all } x\\
&\Rightarrow c \text{ is a } \mu_\gamma\text{-units}
\end{aligned}$$

Similar if $(a / c)_{\mu_\gamma}$ then we can show that b a μ_γ-unit. Hence a is μ_γ-irreducible.

Theorem 5.10. Let $a \in M \setminus \mu_0$ and a is not a μ_γ-unit. Then the element a is μ_γ-irreducible if and only if the ideal $I(a, \gamma, \mu)$ is maximal among all ideals $I(b, \gamma, \mu)$, where $b \in M$ and $\mu(a) \neq \mu(b)$.

Proof. (1) Suppose a is μ_γ-irreducible. Let $I(a, \gamma, \mu) \subseteq I(b, \gamma, \mu) \neq M$, for some $b \in M$ with $\mu(b) \neq \mu(a)$. Now $a \in I(a, \gamma, \mu) \subseteq I(b, \gamma, \mu)$ and so $\mu(a) = \mu(c\gamma b)$, for some $c \in M \setminus \mu_0$. Now if $\mu(a) = \mu(c)$ then $\mu(c) = \mu(c\gamma b)$, which implies $\mu(c\gamma x) = \mu(c\gamma b\gamma x)$, for all $x \in M$. Now M is without μ_γ-divisor of zero and $\mu(c) \neq \mu(0_M)$, so $\mu(x) = \mu(b\gamma x)$ for all $x \in M$. Hence $I(b, \gamma, \mu) = M$, which is not the case. Hence $\mu(a) \neq \mu(c)$. As a is μ_γ-irreducible so either b is a μ_γ-unit or c is a μ_γ-unit. Since $I(b, \gamma, \mu) \neq M$ so by Proposition 4.9, we find that b is not a μ_γ-unit. Hence c is a μ_γ-unit. So there exists $u \in M \setminus \mu_0$ such that $\mu(c\gamma u\gamma x) = \mu(u\gamma c\gamma x) = \mu(x)$, for all $x \in M$. Thus $\mu(b) = \mu(c\gamma u\gamma b)$. Again $\mu(a) = \mu(b\gamma c)$ implies $\mu(a\gamma u) = \mu(b)$. Hence $b \in I(a, \gamma, \mu)$ and so $I(b, \gamma, \mu) \subseteq I(a, \gamma, \mu)$. Consequently, $I(b, \gamma, \mu) = I(a, \gamma, \mu)$. Thus $I(a, \gamma, \mu)$ is maximal.

Conversely, assume $I(a, \gamma, \mu)$ is maximal. Assume that $\mu(a) = \mu(c\gamma d)$, where $c, d \in M$ and $\mu(a) \neq c$, $\mu(a) \neq \mu(d)$. Then $a \in I(d, \gamma, \mu)$ and so $I(a, \gamma, \mu) \subseteq I(d, \gamma, \mu)$. Hence by our hypothesis either $I(a, \gamma, \mu) = I(d, \gamma, \mu)$ or $I(d, \gamma, \mu) = M$. If $I(a, \gamma, \mu) = I(d, \gamma, \mu)$, then $d \in I(d, \gamma, \mu) = I(a, \gamma, \mu)$. Therefore $\mu(d) = \mu(m\gamma a)$, for some $m \in M$. This gives $\mu(c\gamma d) = \mu(c\gamma m\gamma a)$. Thus we have $\mu(a) = \mu(c\gamma m\gamma a)$ and so $\mu(a\gamma(x - c\gamma m\gamma x)) = \mu(0_M)$, for all $x \in M$. As M is without μ_γ-divisor of zero and $\mu(a) \neq \mu(0_M)$, we have $\mu(x) = \mu(c\gamma m\gamma x)$, for all $x \in M$. This shows that c is a μ_γ-unit. If $I(d, \gamma, \mu) = M$, then $\mu(d) = \mu(d\gamma m)$, for some $m \in M$. Again $\mu(m) = \mu(d\gamma y)$, for some $y \in M$. Therefore $\mu(d) = \mu(d\gamma m) = \mu(d\gamma d\gamma y)$. Hence $\mu(d\gamma x) = \mu(d\gamma d\gamma y\gamma x)$. Thus $\mu(x) = \mu(d\gamma y\gamma x)$. This shows that d is a μ_γ-unit.

Theorem 5.11. Let $a \in M \setminus \mu_0$ and a is not a μ_γ-unit. Then a is μ_γ-prime if and only if for $x, y \in M$, $\gamma_1 \in \Gamma$, $x\gamma_1 y \in I(a, \gamma, \mu)$ implies either $x \in I(a, \gamma, \mu)$ or $y \in I(a, \gamma, \mu)$, where $\mu(a) \neq \mu(x)$, $\mu(a) \neq \mu(y)$.

Proof. Let a be μ_γ-prime and $x, y \in M$, $\gamma_1 \in \Gamma$, $x\gamma_1 y \in I(a, \gamma, \mu)$, where $\mu(a) \neq \mu(x)$, $\mu(a) \neq \mu(y)$. Then $\mu(x\gamma_1 y) = \mu(a\gamma m)$, for some $m \in M$. Now this implies that $(a / x\gamma_1 y)_{\mu_\gamma}$. As a is μ_γ-prime so either $(a / x)_{\mu_\gamma}$ or $(a / y)_{\mu_\gamma}$. If $(a / x)_{\mu_\gamma}$, there exists $x_1 \in M$ such that $\mu(x) = \mu(a\gamma x_1)$, which implies $x \in I(a, \gamma, \mu)$. Similarly if $(a / y)_{\mu_\gamma}$ then there exists $y_1 \in M$ such that $\mu(y) = \mu(a\gamma y_1)$, which implies $y \in I(a, \gamma, \mu)$

Conversely, let for $x, y \in M$, $\gamma_1 \in \Gamma$, $x\gamma_1 y \in I(a, \gamma, \mu)$ implies either $x \in I(a, \gamma, \mu)$ or $y \in I(a, \gamma, \mu)$, where $\mu(a) \neq \mu(x)$, $\mu(a) \neq \mu(y)$. We have to prove a is μ_γ-prime. Let $(a / x\gamma_1 y)_{\mu_\gamma}$, where $x, y \in M$, $\gamma_1 \in \Gamma$, $\mu(a) \neq \mu(x)$, $\mu(a) \neq \mu(y)$. Then $\mu(x\gamma_1 y) = \mu(a\gamma m)$, for some $m \in M$. Thus $x\gamma_1 y \in I(a, \gamma, \mu)$. Now from given condition either $x \in I(a, \gamma, \mu)$ or $y \in I(a, \gamma, \mu)$. If $x \in I(a, \gamma, \mu)$, then $\mu(x) = \mu(a\gamma m_1)$, for some $m_1 \in M$. Thus $(a / x)_{\mu_\gamma}$. Similarly if $y \in I(a, \gamma, \mu)$, then $\mu(y) = \mu(a\gamma m_2)$, for some $m_2 \in M$. Thus $(a / y)_{\mu_\gamma}$. This proves that a is μ_γ-prime.

6. IMAGES AND INVERSE IMAGES UNDER Γ-RING HOMOMORPHISMS

Definition 6.1. Let f be a mapping from a Γ-ring M into a Γ-ring M' and let μ be a fuzzy subset of M. Then μ is called f-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$, where $x, y \in M$.

Proposition 6.2. Let f be a homomorphism of a Γ-ring M into a Γ-ring M'. Let σ be a translational invariant fuzzy sbset of M'. Then $f^{-1}(\sigma)$ is a translational invariant fuzzy subset of M.

Proof. Let $a, b \in M$ and $f^{-1}(\sigma)(a) = f^{-1}(\sigma)(b)$. Then $\sigma(f(a)) = \sigma(f(b))$. Let $m \in M$ and $f(m) = m' \in M'$. Since σ is translational invariant fuzzy subset of M' and $\sigma(f(a)) = \sigma(f(b))$, we have $\sigma(f(a) + m') = \sigma(f(b) + m')$ and $\sigma(f(a)\gamma m') = \sigma(f(b)\gamma m')$, $\sigma(m'\gamma f(a)) = \sigma(m'\gamma f(b))$. Now $\sigma(f(a) + m') = \sigma(f(b) + m')$ implies $\sigma(f(a) + f(m)) = \sigma(f(b) + f(m))$ and so $\sigma(f(a + m)) = \sigma(f(b + m))$. Hence $f^{-1}(\sigma)(a + m) = f^{-1}(\sigma)(b + m)$. On the other hand, from $\sigma(f(a)\gamma m') = \sigma(f(b)\gamma m')$ and $\sigma(m'\gamma f(a)) = \sigma(m'\gamma f(b))$, we get $\sigma(f(a)\gamma f(m)) = \sigma(f(b)\gamma f(m))$ and $\sigma(f(m)\gamma f(a)) = \sigma(f(m)\gamma f(b))$. So $\sigma(f(a\gamma m)) = \sigma(f(b\gamma m)$ and $\sigma(f(a\gamma m)) = \sigma(f(b\gamma m))$. Thus we have $f^{-1}(\sigma)(a\gamma m) = f^{-1}(\sigma)(b\gamma m)$ and $f^{-1}(\sigma)(m\gamma a) = f^{-1}(\sigma)(m\gamma b)$, for all $a, b \in M$, $\gamma \in \Gamma$. This proves that $f^{-1}(\sigma)$ is translational fuzzy subset of M.

Proposition 6.3. Let f be a homomorphism of a Γ-ring M onto a Γ-ring M'. Let μ be a translational invariant fuzzy subset of M. Then if μ is f invariant, then $f(\mu)$ is a translational invariant fuzzy subset of M'.

Proof. Suppose μ is f-invariant. Then for all $x, y \in M$, $f(x) = f(y)$ implies that $\mu(x) = \mu(y)$. As f is onto, for $a \in M'$, $f(\mu)(a) = \sup\{\mu(m) : m \in M, f(m) = a\}$. Let $x, y \in M$ and $f(x) = a = f(y)$. Then $f(x) = f(y)$ and so $\mu(x) = \mu(y)$. Hence $f(\mu)(a) = \mu(x)$, where $x \in M$ and $f(x) = a$. Thus for all $a \in M'$, $f(\mu)(a) = \mu(x)$, where $x \in M$ and $f(x) = a$. Now let $a, b \in M'$ and $f(\mu)(a) = f(\mu)(b)$. Then $\mu(x) = \mu(y)$, where $x, y \in M$ such that $f(x) = a$, $f(y) = b$. Let $c \in M'$ be such that $f(z) = c$ for $c \in M$. Then $a + c = f(x) + f(z) = f(x + z)$ and $b + c = f(y) + f(z) = f(y + z)$. Hence $f(\mu)(a+c) = \mu(x+z)$ and $f(\mu)(b+c) = \mu(y+z)$. Again for $\gamma \in \Gamma$, $c\gamma a = f(z)\gamma f(x) = f(z\gamma x)$, $b\gamma c = f(y)\gamma f(z) = f(y\gamma z)$ and $c\gamma b = f(z)\gamma f(y\} = f(z\gamma y)$. Hence $f(\mu)(a\gamma c) = \mu(x\gamma z)$, $f(\mu)(c\gamma a) = \mu(z\gamma x)$, $f(\mu)(b\gamma c) = \mu(y\gamma z)$ and $f(\mu)(c\gamma b) = \mu(z\gamma y)$. Since μ is translational invariant, $\mu(x + z) = \mu(y + z)$, $\mu(x\gamma z) = \mu(y\gamma z)$ and $\mu(z\gamma x) = \mu(z\gamma y)$. Hence $f(\mu)(a + c) = f(\mu)(b + c)$, $f(\mu)(a\gamma c) = f(\mu)(b\gamma c)$ and $f(\mu)(c\gamma a) = f(\mu)(c\gamma b)$. Thus if $a, b \in M'$, $\gamma \in \Gamma$ and $f(\mu)(a) = f(\mu)(b)$, then $f(\mu)(a + c) = f(\mu)(b + c)$, $f(\mu)(a\gamma c) = f(\mu)(b\gamma c)$ and $f(\mu)(c\gamma a) = f(\mu)(c\gamma b)$ for all $c \in M'$. Hence $f(\mu)$ is a translational invariant fuzzy subset of M'.

Theorem 6.4. Let f be a homomorphism of a Γ-ring M onto a Γ-ring M' and μ be a translational invariant fuzzy subset of M. If μ is f-invariant then $f(I(a, \gamma, \mu) = I(f(a), \gamma, f(\mu))$, for $a \in M$, $\gamma \in \Gamma$.

Proof. Supose μ is f-invairant. Let $y \in I(f(a), \gamma, f(\mu))$. Then $f(\mu)(y) = f(\mu)(s\gamma f(a))$ for some $s \in M'$. Since $y, s \in M'$ and f is onto, there exists $x, r \in M$ such that $f(x) = y$ and $f(r)$

$= s$. Thus $f(\mu)(f(x)) = f(\mu)(f(r)\gamma f(a)) = f(\mu)f(r\gamma a)$. Since μ is translational invariant, by Proposition 5.2 we get $f(\mu)(f(x)) = \mu(x)$ and $f(\mu)(f(r\gamma a) = \mu(r\gamma a)$. Thus $\mu(x) = \mu(r\gamma a)$, which implies $x \in I(a, \gamma, \mu)$ and so $f(x) \in f(I(a, \gamma, \mu))$ that is $y \in f(I(a, \gamma, \mu))$. Hence $I(f(a), \gamma, f(\mu)) \subseteq f(I(a, \gamma, \mu)$. Again let $y \in f(I(a, \gamma, \mu))$. Then there exists $x \in I(a, \gamma, \mu)$ such that $f(x) = y$. Also, $x \in I(a, \gamma, \mu)$ implies $\mu(x) = \mu(a\gamma r)$ for some $r \in M$. Now,

$$\begin{aligned} f(\mu)(y) &= \sup\{\mu(x) : x \in f^{-1}(y)\} \\ &= \mu(x), \text{ since } \mu \text{ is } f\text{-invariant} \\ &= \mu(a\gamma r). \end{aligned}$$

Also if $f(r) = s$ we have $f(\mu)(f(a)\gamma s) = f(\mu)(f(a)\gamma f(r)) = f(\mu)f(a\gamma r) = \sup\{\mu(x')$, such that $x' \in f^{-1}(f(a\gamma r))\} = \mu(a\gamma r)$, since μ is f-invariant. Thus $f(\mu)(y) = f(\mu)(f(a\gamma s)\lambda)$ which implies that $y \in I(f(a), \gamma, f(\mu))$. Hence $f(I(a, \gamma, \mu)) \subseteq I(f(a), \gamma, f(\mu))$, $a \in M$, $\gamma \in \Gamma$. Consequently, $f(I(a, \gamma, \mu)) = I(f(a), \gamma, f(\mu))$, for $a \in M$, $\gamma \in \Gamma$.

Proposition 6.5. Let f be a homomorphism of a Γ-ring M onto a Γ-ring M'. Let σ be a translational invariant fuzzy subset of M'. Let $a' \in M'$. Then for all $a, b \in f^{-1}(a')$, $I(a, \gamma, f^{-1}(\sigma) = I(b, \gamma, f^{-1} \supset (\sigma))$, provided $f^{-1}(a')$ contains more than one element.

Proof. Let $x \in I(a, \gamma, f^{-1}(\sigma))$. Then $f^{-1}(\sigma)(x) = f^{-1}(\sigma)(r\gamma a)$, for some $r \in M$ and so $f^{-1}(\sigma)(x) = \sigma(f(r\gamma a))$. Thus $f^{-1}(\sigma)(x) = \sigma(f(r)\gamma f(a))$. Since $a, b \in f^{-1}(a')$, $f(a) = f(b) = a'$ and hence we have $f^{-1}(\sigma)(x) = \sigma(f(r)\gamma f(b)) = \sigma(f(r\gamma b)) = f^{-1}(\sigma)(r\gamma b)$. This shows that $x \in I(b, \gamma, f^{-1}(\sigma))$. Hence $I(a, \gamma, f^{-1}(\sigma)) \subseteq I(b, \gamma, f^{-1}(\sigma))$. Now let $y \in I(b, \gamma, f^{-1}(\sigma))$. Then $f^{-1}(\sigma)(y) = f^{-1}(\sigma)(r'\gamma b)$, for some $r' \in M$ and so $f^{-1}(\sigma)(y) = \sigma(f(r'\gamma b))$. Thus $f^{-1}(\sigma)(y) = \sigma(f(b)\gamma f(r'))$. Since $a, b \in f^{-1}(a')$, $f(a) = f(b) = a'$ and hence we have $f^{-1}(\sigma)(y) = \sigma(f(a)\gamma f(r')) = \sigma(f(a\gamma r')) = f^{-1}(\sigma)(a\gamma r')$. This shows that $y \in I(a, \gamma, f^{-1}(\sigma))$. Hence $I(b, \gamma, f^{-1}(\sigma)) \subseteq I(a, \gamma, f^{-1}(\sigma))$. This proves that $I(a, \gamma, f^{-1}(\sigma) = I(b, \gamma, f^{-1}(\sigma))$, for all $a, b \in f^{-1}(a')$.

Theorem 6.6. Let f be an isomomorphism of a Γ-ring M onto a Γ-ring M'. Let σ be a translational invariant fuzzy subset of M'. Then $I(f^{-1}(y), \gamma, f^{-1}(\sigma) = f^{-1}(I(y, \gamma, \sigma)$, for all $y \in M'$, $\gamma \in \Gamma$.

Proof. Let $x \in I(f^{-1}(y), \gamma, f^{-1}(\sigma))$. Then,

$$\begin{aligned} f^{-1}(\sigma)(x) &= f^{-1}(\sigma)(f^{-1}(y)\gamma r), \text{ for some } r \in M \\ &= f^{-1}(\sigma)(f^{-1}(y)\gamma f^{-1}(s)), \text{ where } s \in M' \text{ such that } f(r) = s \end{aligned}$$

$$\begin{aligned}\Rightarrow \sigma(f(x)) &= f^{-1}(\sigma)(f^{-1}(y\gamma s)), \text{ since } f \text{ is bijective}\\ &= \sigma(f(f^{-1}(y\gamma s)))\\ &= \sigma(y\gamma s)\\ \Rightarrow f(x) &\in I(y, \gamma, \sigma)\\ \Rightarrow x &\in f^{-1}(I(y, \gamma, \sigma)).\end{aligned}$$

Hence $I(f^{-1}(y), \gamma, f^{-1}(\sigma)) \subseteq f^{-1}(I(y, \gamma, \sigma))$ for all $y \in M'$. Again let $a \in f^{-1}(I(y, \gamma, \sigma))$ then $f(a) \in I(y, \gamma, \sigma)$ implies $\sigma(f(a)) = \sigma(y\gamma s)$, for some $s \in M'$. Also $y, s \in M'$ and f is onto implies that there exist $x, r \in R$ such that $f(x) = y$ and $f(r) = s$. Now $\sigma(f(a)) = \sigma(y\gamma s)$ implies $\sigma(f(a)) = \sigma(f(x)\gamma f(r)) = \sigma(f(x\gamma r))$ implies that $f^{-1}(\sigma)(a) = f^{-1}(\sigma)(x\gamma r) = f^{-1}(\sigma)(f^{-1}(y)\gamma r)$ which implies $a \in I(f^{-1}(y), \gamma, f^{-1}(\sigma))$. Thus $f^{-1}(I(y, \gamma, \sigma) \subseteq I(f^{-1}(y), \gamma, f^{-1}(\sigma))$, for all $y \in M'$, $\gamma \in \Gamma$. This proves that $I(f^{-1}(y), \gamma, f^{-1}(\sigma) = f^{-1}(I(y, \gamma, \sigma)$, for all $y \in M'$, $\gamma \in \Gamma$.

REFERENCES

1. Barnes, W.E. ; *On the Γ-rings of Nobusawa,* Pacific J. Math., 18(1996), 411-422.
2. Coppage, W.E. and Luh, J.: *Radicals of Γ-rings,* J. Math. Soc. Japan, 23(1971), 40-52.
3. Dutta, T.K. and Chanda, T.: *Structures of Fuzzy Ideals of Γ-ring,* Bull. Malaysian Math. Sci. Soc., 28(1)(2005), 9-18.
4. Hong, S.M. and Jun, Y.B.: *A note on fuzzy ideals in gamma-rings,* Bull. Homam Math. Soc., 12(1995), 39-48.
5. Mukherjee, T.K. and Sen, M.K.: *On fuzzy ideals of a ring I,* Fuzzy Sets and Systems, 21(1987), 99-104.
6. Jun, Y.B. and Lee, C.Y.: *Fuzzy Γ-rings,* Pusan Kyongnam Math. J. (presently, East Asian Math. J.), 8(2)(1992), 163-170.
7. Jun, Y.B. and Lee, C.Y.: *Fuzzy prime ideals in Γ-rings,* Pusan Kyongnam Math. J. (presently, East Asian Math. J.), 9(1)(1993), 105-111.

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FURTHER INVESTIGATIONS ON *P*-PRIME IDEALS

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ABSTRACT : The study of p-prime ideals in a semiring was initiated in [1]. The present paper investigates some more properties of p-prime ideals in connection with various notions in semiring related to its p-ideals. Here we invoke the notion of p-irreducible ideals in a semiring [8] and provide some necessary and sufficient conditions for a p-ideal of a semiring to be strongly p-irreducible. The p-regular semirings are characterised in terms of p-prime ideals and p-irreducible ideals of it. Further, the structure of periodically finitely generated p-radical of an ideal in a semiring is determined. Finally, the p-noetherian semirings are characterised in terms of their p-prime ideals.

Key words : p-ideal, p-prime ideal, p-semiprime ideal, p-irreducible ideal.
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1. INTORDUCTION AND PRELIMINARIES

A semiring is a non-empty set R together with two binary operations, called "addition", +, and "multiplication" (usually denoted by juxtaposition), such that R is multiplicatively a semigroup and additively a commutative semigroup and the multiplication is distributed accross the addition both from the left and from the right. A semiring is said to be *commutative* if it is multiplicatively commutative. A semiring R is said to have an identity, if there exists $1_R \in R$ such that, $1_R x = x 1_R = x$ for each $x \in R$. The zero element of R, denoted by 0_R (provided exists in R), is called an *absorbing zero* if $a0_R = 0_R a = 0_R$ for all $a \in R$. An ideal [3] I of a semiring R is called a k-ideal [5] of R if for any $x, y \in R$, $x + y \in I$ and $x \in I \Rightarrow y \in I$. The smallest k-ideal containing an ideal I of R, called the k-closure of I is given by $\bar{I} = \{x \in R : x + y = z \text{ for some } y, z \in I\}$.

J. S. Golan [3] defined prime ideals in a semiring and discussed several results on prime ideals in semirings alongside the existing literature of ring theory. P. Mukhopadhyay and S. Ghosh [4] introduced the idea of p-ideals in semirings. We recall the following definitions and results from [4, 6] that will be useful in the sequel. Throughout this paper N represents the set of natural numbers.

Definition 1.1. An ideal I of a semiring R is called a *p-ideal* if for some $x \in R$, $n \in \mathrm{N}$,

$$a + nx = (n + 1)x \text{ and } a \in I \text{ implies } x \in I.$$

For a semiring R, the set of all periodic elements in R is defined as $P^+(R) = \{a \in S \mid na = (n+1)a$, for some $n \in \mathbb{N}\}$ [4]. Note that if R has an absorbing zero then, $P^+(R) \subseteq I$, for all p-ideals I. For any subsemiring S of a semiring R, it is defined in [5] that,

$$\hat{S} = \{x \in R \mid a + nx = (n+1)x, \text{ for some } n \in \mathbb{N},\ a \in S\}.$$

Proposition 1.2. *For any two ideals I, J of a semiring R, $\hat{I}$ is a p-ideal of R such that $\hat{\hat{I}} = \hat{I}$; $I \subseteq \hat{I}$; if $I \subseteq J$ then $\hat{I} \subseteq \hat{J}$; indeed, $\hat{I}$ is the smallest p-ideal of R containing I.*

Definition 1.3. A semiring R is called p-regular [7] if for each $a \in R$ there exists $b \in R$ such that

$$aba + na = (n+1)a \text{ for some } n \in \mathbb{N}.$$

Proposition 1.4. [4] *A semiring R with 1_R is p-regular if and only if for every right p-ideal A and left p-ideal B, we have $A \cap B = \widehat{AB}$.*

2. DIVISORS OF $P^+(R)$ AND p-PRIME IDEALS

In [1], U. Dasgupta and P. Mukhopadhyay initiated the investigation of that class of prime (semiprime) ideals in a semiring which are also p-ideals and developed the corresponding theory of p-prime (resp. p-semiprime) ideals.

Definition 2.1. An ideal I of a semiring R is called a *p-prime* (*p-semiprime*) *ideal* if it is a p-ideal as well as a prime (resp. semiprime) ideal of R.

Following results related to p-prime ideals in a semiring were among those proved in [1].

Proposition 2.2. *A proper ideal I of a semiring R is p-prime (p-semiprime) if and only if for any p-ideals H and K of R, $HK \subseteq I$ (resp. $H^2 \subseteq I$) implies $H \subseteq I$ or $K \subseteq I$ (resp. $H \subseteq I$).*

Proposition 2.3. *For a proper p-ideal I of a semiring R, the following conditions are equivalent:*

(i) *I is a p-prime ideal.*

(ii) *$I^c = R \setminus I$ is a weak m-system.*

A non-empty set S of a semiring R is called a *weak-m-system* [1], if for any $a, b \in S$, $\widehat{aRb} \cap S \neq \phi$. Clearly any multiplicative set is a weak m-system.

Proposition 2.4. *If A is a weak m-system of a semiring R and if I is a p-ideal of R, which is maximal among all those p-ideals of R disjoint from A, then I is a p-prime ideal of R.*

As a consequence of the above proposition, one may have the following :

Proposition 2.5. *A p-ideal, maximal among all the p-ideals of a semiring R, is a p-prime ideal.*

We now consider the notion of divisors of $P^+(R)$ for a semiring R, which virtually ushers in the idea of divisors of zero in semiring. The study on divisors of $P^+(R)$ is extensively made in [1], [4] and [6].

Definition 2.6. [4] Let R be a semiring. An element $a \in R \setminus P^+(R)$ is called a left (resp. right) divisor of $P^+(R)$ in R if there exists $b \in R \setminus P^+(R)$ such that $ab \in P^+(R)$ (resp. $ba \in P^+(R)$). An element $a \in R$ is called a divisor of $P^+(R)$ if it is both a left and a right divisor of $P^+(R)$.

We only remark here that a commutative semiring R has no divisor of $P^+(R)$ if and only if $P^+(R)$ is a p-prime ideal of R. That the set of all divisors of $P^+(R)$ in a commutative semiring R can be covered by a class of p-prime ideals in R is established in the following proposition.

Proposition 2.7. *Let R be a non-trivial commutative semiring with 1_R and absorbing zero. Then the set consisting of all periodic elements and all divisors of $P^+(R)$ is a union of some p-prime ideals of R.*

Proof. Let A be the set of all $x \in R \setminus P^+(R)$ such that, for any $y \in R$, $xy \in P^+(R) \Rightarrow y \in P^+(R)$. Then $A \neq \phi$, since $1_R \in A$. For any $x_1, x_2 \in A$ and $r \in R$, $r(x_1x_2) \in P^+(R) \Rightarrow (rx_1)x_2 \in P^+(R) \Rightarrow rx_1 \in P^+(R) \Rightarrow r \in P^+(R) \Rightarrow x_1x_2 \in A$. Thus A is a multiplicative set and hence a weak m-system. Clearly here the set A^c consists precisely the periodic elements and the divisors of $P^+(R)$. Let a be any divisor of $P^+(R)$. Then $\widehat{aR}$ is a p-ideal in which every element is either periodic or else a divisor of $P^+(R)$. In fact, for any $x \in \widehat{aR} \setminus P^+(R)$, there exists $r \in R$ such that $ar + nx = (n + 1)x$ for some $n \in \mathbb{N}$. Since a is a divisor of $P^+(R)$, so there exists $b \in R \setminus P^+(R)$ such that $ab \in P^+(R)$. Then $abr + nxb = (n + 1)xb$ which implies that $xb \in \widehat{P^+(R)} = P^+(R)$ (since $ab \in P^+(R) \Rightarrow abr \in P^+(R)$). Hence x is a divisor of $P^+(R)$. Thus $\widehat{aR} \subseteq A^c$. So, by Zorn's lemma, we have a maximal p-ideal P_a containing $\widehat{aR}$, which is disjoint from the weak m-system A. By proposition 2.4, P_a is a p-ideal such that $P_a \subseteq A^c$. Moreover $a \in P_a$ and $P^+(R) \subseteq P_a$ (since R has absorbing zero). Thus $A^c = \cup P_a$.

3. *p*-IRREDUCIBLE AND *p*-PRIME IDEALS

A p-ideal I of a semiring R is called an *irreducible* p-ideal (or interchangably *p-irreducible* ideal) [8] of R if $I = H \cap K \Rightarrow I = H$ or $I = K$ for any p-ideals H and K of R. The p-ideal I is called strongly p-irreducible if $H \cap K \subseteq I \Rightarrow H \subseteq I$ or $K \subseteq I$ for any p-ideals H and K or R. This is clear from these definitions that every strongly p-irreducible ideal in a semiring is p-irreducible. M. Shabir and S. Hussain [8] characterized fully p-prime semirings (i.e., the semirings in which every p-ideal is a p-prime ideal) in terms of p-irreducible ideals. They have also proved [8] that every p-ideal of a semiring R is the intersection of all p-irreducible *ideals* *containing* it. We now present two necessary and sufficient conditions for a p-ideal in a semiring to be a strongly p-irreducible ideal. Towards this, we first introduce the notion of weak i-system in a semiring.

Definition 3.1. A non-empty subset S of a semiring R is called a *weak i-system* if for any $a, b \in S$, $\widehat{(a)} \cap \widehat{(b)} \cap S \neq \phi$.

Proposition 3.2. *The following conditions on a p-ideal I of a semiring R are equivalent:*

(1) *I is strongly p-irreducible,*

(2) $\widehat{(a)} \cap \widehat{(b)} \subseteq I \Rightarrow$ *either* $a \in I$ *or* $b \in I$ *for any* $a, b \in R$.

(3) $R \setminus I$ *is a weak i-system.*

Proof. (1) $\Rightarrow$ (2) follows immediately from definitions.

(2) $\Rightarrow$ (3): If $a, b \in R \setminus I$ and $\widehat{(a)} \cap \widehat{(b)} \cap (R \setminus I) = \phi$, then $\widehat{(a)} \cap \widehat{(b)} \subseteq I$ and so, by (2) $a \in I$ or $b \in I$, which yields a contradiction.

(3) $\Rightarrow$ (1): Let H and K be any two p-ideals of R such that $H \nsubseteq I$ and $K \nsubseteq I$. Then there are $a \in H \setminus I$ and $b \in K \setminus I$. Then by (3), $\widehat{(a)} \cap \widehat{(b)} \cap (R \setminus I) \neq \phi$. So there exists $c \in \widehat{(a)} \cap \widehat{(b)} \subseteq H \cap K$ such that $c \notin I$, whence $H \cap K \nsubseteq I$. Hence I is a p-irreducible ideal of R.

That for a p-ideal I, being semiprime and strongly p-irreducible is necessary and sufficient condition for I to be a prime ideal is established in [8]. Thus a p-prime ideal in a semiring is always a p-irreducible ideal. But a p-irreducible ideal in a semiring may not be a p-prime ideal. However, the last proposition of this section shows that these two notions of p-primeness and p-irreducibility of a p-ideal merge with each other in a commutative p-regular semiring with unity. Before going to that we would like to restate the following two propositions that were proved in [1].

Proposition 3.3. *Intersection of any class of p-prime ideals of a semiring R is a p-semiprime ideal of R.*

Proposition 3.4. *A commutative semiring R is p-regular if and only if every p-ideal is semiprime.*

Proposition 3.5. *A commutative semiring R with* 1_R *is p-regular if and only if every p-irreducible ideal of R is a p-prime ideal.*

Proof. Suppose that the commutative semiring R with 1_R is p-regular. Let I be a p-irreducible ideal of R. If possible, let $R \setminus I$ be not a weak m-system in R. Then, there exist $a, b \in R \setminus I$ such that $\widehat{abR} \subseteq I$.

We consider two p-ideals $H = \widehat{I+\widehat{(a)}}$ and $K = \widehat{I+\widehat{(b)}}$. Then for any $x \in H$ and $y \in K$, there exist $x_1 \in I + \widehat{(a)}$ and $y_1 \in I + \widehat{(b)}$ such that for some $n, m \in \mathbb{N}$

$$x_1 + nx = (n+1)x \ \ldots(1) \quad \text{and} \quad y_1 + my = (m+1)y \ \ldots(2)$$

and also $x_1 = \alpha + a_1$ ...(3) and $y_1 = \beta + b_1$ ...(4) for some $\alpha, \beta \in I$ and $a_1 \in (a)$ and $b_1 \in (b)$. Clearly then, for some $p, q \in \mathbb{N}$ and $z, t \in R$,

$$az + pa_1 = (p+1)a_1 \ \ldots \ (5) \quad \text{and} \quad bt + qb_1 = (q+1)b_1 \ \ldots \ (6).$$

Hence from (5) $abz + pa_1b = (p+1)a_1b$

$\Rightarrow abzt + pa_1bt = (p+1)a_1bt$

$\Rightarrow abzt + pa_1bt + pqa_1b_1 = (p+1)a_1bt + pqa_1b_1$

$\Rightarrow abzt + pa_1(bt + qb_1) = pa_1(bt + qb_1) + a_1bt$

$\Rightarrow abzt + p(q+1)a_1b_1 = p(q+1)a_1b_1 + a_1bt$ (from (6))

$\Rightarrow abzt + p(q+1)a_1b_1 = p(q+1)a_1b_1 + a_1bt$

$\Rightarrow abzt + p(q+1)a_1b_1 + qa_1b_1 = p(q+1)a_1b_1 + a_1bt + qa_1b_1$

$\Rightarrow abzt + (pq + p)a_1b_1 + qa_1b_1 = p(q+1)a_1b_1 + a_1(bt + qb_1)$

$\Rightarrow abzt + (pq + q + q)a_1b_1 = p(q+1)a_1b_1 + (q+1)a_1b_1$

$\Rightarrow abzt + (pa + p + q)a_1b_1 = (pq + p + q + 1)a_1b_1$

$\Rightarrow abzt + (pq + p + q)a_1b_1 + (pq + p + q + 1)(\alpha b_1 + \beta a_1 + \alpha\beta) = (pq + p + q + 1)a_1b_1$
$+ (pq + p + q + 1)(\alpha b_1 + \beta a_1 + \alpha\beta)$

$\Rightarrow abzt + \alpha b_1 + \beta a_1 + \alpha\beta + (pq + p + q)(a_1b_1 + \alpha b_1 + \beta a_1 + \alpha\beta) = (pq + p + q + 1)(a_1b_1$
$+ \alpha b_1 + \beta a_1 + \alpha\beta)$

$\Rightarrow (abzt + \alpha b_1 + \beta a_1 + \alpha\beta) + (pq + p + q)x_1y_1 = (pq + p + q + 1)x_1y_1$ (by (3) and (4)).

But here α, β, $abzt \in I$ (since $\widehat{abR} \subseteq I$); so, $abzt + \alpha b_1 + \beta a_1 + \alpha\beta \in I$. Hence, $x_1y_1 \in \hat{I} = I$ (since I is a p-ideal). Then from (1), $x_1y_1 + nxy_1 = (n+1)xy_1 \Rightarrow xy_1 \in \hat{I} = I$ (since $x_1y_1 \in I$). Hence from (2), $xy_1 + mxy = (m+1)xy \Rightarrow xy \in \hat{I} = I$ (since $xy_1 \in I$). Thus for any $x_i \in H$ and $y_i \in K$, the finite sum $\sum x_iy_i \in I$, i.e., $HK \subseteq I$. Clearly here $I \subseteq HK$. So, $HK = I$ and hence, $\widehat{HK} = I$ (since I is a p-ideal). Since R is a p-regular semiring with 1_R, so by proposition 1.4, $H \cap K = \widehat{HK}$, whence $H \cap K = I$ and hence either $\widehat{I+\widehat{(a)}} = H = I$ or

$\widehat{I+\widehat{(b)}} = K = I$ (since I is a p-irreducible ideal of R), i.e., either $a \in I$ or $b \in I$ which is a contradiction. Hence, $R \setminus I$ is a weak m-system in R. Thus by proposition 2.3, I is a p-prime ideal of R.

Conversely assume that every p-irreducible ideal of R is p-prime. Let I be any p-ideal of R. Then I is the intersection of all p-irreducible ideals of R containing I. Thus here I is the intersection of a class of p-prime ideals of R. Hence by proposition 3.3, I is a p-semiprime ideal of R. Hence by proposition 3.4, R is a p-regular semiring.

4. PERIODICALLY GENERATED IDEALS AND *p*-PRIME IDEALS

A p-ideal I of a semiring R is said to be periodically generated by a non-empty subset A of R if $I = \widehat{(A)}$ where (A) denotes the ideal of the semiring R generated by A. Thus in other words, the periodically generated ideal I generated by $A \in P^*(R)$ is the smallest p-ideal of R containing A. A p-ideal of R is said to be periodically finitely generated if it is periodically generated by a finite set. It is worth noticing here that the p-ideal $P^+(R)$ of all periodic elements of a semiring R with absorbing zero 0_R is periodically finitely generated by $\{0_R\}$. It is also clear that in a semiring, a p-ideal which is finitely generated as an ideal is a periodically finitely generated p-ideal. Following are some examples of periodically finitely generated p-ideals in a semiring which are not finitely generated as an ideal.

Example 4.1. For two positive integers $m > 1$ and $n > 1$, we consider

$$S(m,\ n) = \{\overline{x} : x \in Z_0^+, x \le m\} \cup \{\overline{m+1}, \overline{m+2}, \ldots, \overline{m+n}\} \text{ where}$$

$$\overline{x} = \begin{cases} \{x\} & \text{when } x \le m, \\ \{nk + x : k \in Z_0^+\} & \text{when } x \ge m+1, \end{cases}$$

Z_0^+ being the set of all non-negatie integers. Then $S(m, n)$ is a semiring with respact to usual addition and multiplication of classes. Let C denote the interval $[0, 1)$. Consider the set $T(m, n) = S(m, n) \times C$, where we define the operations as follows:

$$(\overline{a}, i) \oplus (\overline{b}, j) = (\overline{a+b}, \max(i, j)) \text{ and}$$

$$(\overline{a}, i) \bullet (\overline{b}, j) = (\overline{ab}, \min(i, j)) \text{ for all } (\overline{a}, i), (\overline{b}, j) \in T(m, n)$$

Then it is routine to check that $(T(m, n), \oplus, \bullet)$ is a semiring with absorbing zero $0_T = (\overline{0}, 0)$.

(a) In the semiring $R = T(6, 3)$, the periodically finitely generated p-ideal $P^+(R) = \{(\overline{x}, i) : x = 0, 3, 4, 9;\ i \in C\}$ of all periodic elements of the semiring R is not finitely generated.

(b) In the semiring $R = T(6, 4)$, $P^+(R) = \{(\overline{x}, i) : x = 0, 4, 8;\ i \in C\}$. Then the periodically finitely generated p-ideal $I = \widehat{(A)}$, where $A = \{(\overline{10}, 0)\}$ is such that $I \neq P^+(R)$ (since $(\overline{10}, 0) \notin P^+(R)$). Also, I is a proper p-ideal of T. In fact, $I = T \Rightarrow (\overline{1}, 1) \in I \Rightarrow$ there exist $p \in Z_0^+$ and $q \in \mathrm{N}$ such that $(\overline{10p}, 0) + q(\overline{1}, 1) = (q+1)(\overline{1}, 1) \Rightarrow \overline{10p} + q\overline{1} = (q+1)\overline{1} \Rightarrow \overline{10p+q} = \overline{(q+1)} \Rightarrow 4k + 10p + q = 4l + q + 1$ for some $k, l \in Z_0^+ \Rightarrow 4\alpha + 10p = 1$, ($\alpha = k - l$) which contradicts the fact that α, p are two integers, since $gcd(4, 10)$ does not divide 1. Clearly here the proper p-ideal I is not finitely generated as an ideal.

Lemma 4.2. *Let I be an ideal of a commutative semiring R such that the p-ideal* $J = \hat{I}$ *be periodically finitely generated. Then* $J = \widehat{(\alpha_1, \alpha_2, \ldots, \alpha_n)}$ *for some* $n \in \mathrm{N}$ *and some* $\alpha_i \in I$, $(i = 1, 2, \ldots, n)$.

Proof. Since $J = \hat{I}$ is periodically finitely generated in the commutative semiring R, so there exist $r_1, r_2, \ldots, r_n \in R$ for some $m \in \mathrm{N}$ such that $J = \widehat{(r_1, r_2, \ldots, r_n)}$. Then for each $i(= 1, 2, \ldots, n)$ $r_i \in \hat{I} \Rightarrow \alpha_i + p_i r_i = (p_i + 1)r_i$ for some $p_i \in \mathrm{N}$ and $\alpha_i \in I$. Again for any $x \in J = \widehat{(r_1, r_2, \ldots, r_n)}$, there exists $q \in \mathrm{N}$ such that $\sum_{i=1}^{n} y_i r_i + qx = (q+1)x$, where for each i, either $y_i \in \mathrm{N}$ or $y_i \in R$. Let $p = max\{q, p_i : i = 1, 2, \ldots, n\}$. Then,

$$\alpha_i + pr_i = (p+1)r_i \ \ldots (1) \text{ and } \sum_{i=1}^{n} y_i r_i + px = (p+1)x \ \ldots (2)$$

Now from (1), $y_i\alpha_i + py_i r_i = (p+1)y_i r_i \Rightarrow \sum_{i=1}^{n} y_i\alpha_i + p\sum_{i=1}^{n} y_i r_i = (p+1)\sum_{i=1}^{n} y_i r_i$

$$\Rightarrow \sum_{i=1}^{n} y_i\alpha_i + p\sum_{i=1}^{n} y_i r_i + p(p+1)x = (p+1)\sum_{i=1}^{n} y_i r_i + p(p+1)x$$

$$\Rightarrow \sum_{i=1}^{n} y_i\alpha_i + p\left(\sum_{i=1}^{n} y_i r_i + px\right) + px = (p+1)\left(\sum_{i=1}^{n} y_i r_i + px\right)$$

$$\Rightarrow \sum_{i=1}^{n} y_i\alpha_i + p(p+1)x + px = (p+1)^2 x \text{ (from (2))} \Rightarrow \sum_{i=1}^{n} y_i\alpha_i + \left(p^2+2p\right)x = (p+1)^2 x$$

$$\Rightarrow \sum_{i=1}^{n} y_i\alpha_i + \left((p+1)^2-1\right)x = (p+1)^2 x \Rightarrow x \in (\alpha_1, \widehat{\alpha_2, \ldots,} \alpha_n) \Rightarrow J \subseteq (\alpha_1, \widehat{\alpha_2, \ldots,} \alpha_n)$$

whence $J = (\alpha_1, \widehat{\alpha_2, \ldots,} \alpha_n)$ (since $(\alpha_1, \alpha_2, \ldots, \alpha_n) \subseteq I$).

Proposition 4.3. *Let R be a commulative semiring with 1_R and absorbing zero and I be a p-ideal which also a k-ideal of R. If I is maximal among all p-ideals which are not periodically finitely generated then I is a p-prime ideal of R.*

Proof. Let $ab \in I$, but $a \notin I$ and $b \notin I$. We consider the p-ideals $\widehat{I+(a)}$ and $\widehat{I+(b)}$. Now, $\widehat{I+(a)} = R \Rightarrow 1_R \in \widehat{I+(a)} \Rightarrow \alpha + xa + n1_R = (n+1)1_R \Rightarrow \alpha b + xab + nb = (n+1)b$ for some $\alpha \in I$, $x \in R$ and $n \in \mathrm{N} \Rightarrow b \in \hat{I} = I$ which is a contradiction. So $\widehat{I+(a)}$ is a proper p-ideal of R. Similarly $\widehat{I+(b)}$ is a proper p-ideal of R. Moreover I is properly contained in both $\widehat{I+(a)}$ and $\widehat{I+(b)}$ (since $a \notin I$, $b \notin I$ and R has absorbing zero). Thus $\widehat{I+(a)}$ and $\widehat{I+(b)}$ are periodically finitely generated p-ideal of R. Hence, by lemma 4.2, there exist α_i, $\beta_j \in I$ and x_i, $y_j \in R$, ($i = 1, 2, \ldots n$, $j = 1, 2, ..m$ for some n, $m \in \mathrm{N}$) such that $\widehat{I+(a)} = (\{\alpha_i \widehat{+x_ia : i}\})$ and $\widehat{I+(b)} = (\{\beta_j \widehat{+y_jb : j}\})$. Consider the p-ideal $J = \{r \in \widehat{R : ra} \in I\}$. Observe that $(\beta_j + y_jb)a = \beta_ja + y_jab \in I \Rightarrow \beta_j + y_jb \in J$ for each j. Then $x \in \widehat{I+(b)}$ $\Rightarrow \sum_{j=1}^{m} r_j\left(\beta_j + y_jb\right) + lx = (l+1)x$ for some $r_j \in R$ and $l \in \mathrm{N} \Rightarrow x \in \hat{J} = J \Rightarrow \widehat{I+(b)}$ $\subseteq J$. Note that $J \neq R$ (since $a \notin I$). Thus we have that $I \subsetneqq \widehat{I+(b)} \subseteq J \subsetneqq R$ and hence by the maximality of I, the p-ideal $J = \{r \in \widehat{R : ra} \in I\}$ of R is periodically finitely generated. But $\{r \in R : ra \in I\}$ is an ideal of R (since the semiring R is commutative). Hence, by lemma 4.2, there exist $j_k \in \{r \in R : ra \in I\}$, ($k = 1, 2, \ldots, l$) such that $J = (\{\widehat{j_k : k}\})$.

Now let $c \in I$ be arbitrary. Then $c \in \widehat{I+(a)}$ $\Rightarrow \sum_{i=1}^{n} s_i\left(\alpha_i + x_ia\right) + pc = (p+1)c$ for some $s_i \in R$ and $p \in \mathrm{N}$ $\Rightarrow \sum_{i=1}^{n} s_i\alpha_i + \left(\sum_{i=1}^{n} s_ix_i\right)a + pc = (p+1)c$ ... (1)

$\Rightarrow \left(pc + \sum_{i=1}^{n} s_i\alpha_i\right) + \left(\sum_{i=1}^{n} s_i x_i\right)a = (p+1)c \Rightarrow \left(\sum_{i=1}^{n} s_i x_i\right)a \in I$ (since I is a k-ideal of R and $pc + \sum_{i=1}^{n} s_i\alpha_i \in I$, $(p+1)c \in I$) $\Rightarrow \sum_{i=1}^{n} s_i x_i \in J = (\widehat{\{j_k : k\}}) \Rightarrow$ there exist $t_k \in R$, $(k = 1, 2, \ldots, l)$ and $q \in \mathbb{N}$ such that $\sum_{k=1}^{l} t_k j_k + q\sum_{i=1}^{n} s_i x_i = (q+1)\left(\sum_{i=1}^{n} s_i x_i\right)$. Assuming without any loss of generality that $p \geq q$ we have

$$\sum_{k=1}^{l} t_k j_k + p\sum_{i=1}^{n} s_i x_i = (p+1)\left(\sum_{i=1}^{n} s_i x_i\right) \quad \ldots (2)$$

Now from (1), $(p+1)\left(\sum_{i=1}^{n} s_i\alpha_i\right) + (p+1)\left(\sum_{i=1}^{n} s_i x_i\right)a + p(p+1)c = (p+1)^2 c$

$$\Rightarrow (p+1)\left(\sum_{i=1}^{n} s_i\alpha_i\right) + \left(\sum_{k=1}^{l} t_k j_k + p\sum_{i=1}^{n} s_i x_i\right)a + p(p+1)c = (p+1)^2 c \text{ (from (2))}$$

$$\Rightarrow \sum_{i=1}^{n} s_i\alpha_i + \left(\sum_{k=1}^{l} t_k j_k\right)a + p\left(\sum_{i=1}^{n} s_i\alpha_i + \left(\sum_{i=1}^{n} s_i x_i\right)a + pc\right) + pc = (p+1)^2 c$$

$$\Rightarrow \sum_{i=1}^{n} s_i\alpha_i + \left(\sum_{k=1}^{l} t_k j_k\right)a + p(p+1)c + pc = (p+1)^2 c \text{ (from (1))}$$

$$\Rightarrow \sum_{i=1}^{n} s_i\alpha_i + \sum_{k=1}^{l} t_k(j_k a) + \left((p+1)^2 - 1\right)c = (p+1)^2 c \Rightarrow c \in \hat{K},$$

where $K = (\alpha_1, \alpha_2, \ldots, \alpha_n, j_1 a, j_2 a, \ldots, j_l a)$ is a finitely generated ideal of R. Thus $I \subseteq \hat{K}$. Since $j_k \in \{r \in R : ra \in I\}$, $(k = 1, 2, \ldots, l)$, we have that $K \subseteq I$, whence $I = \hat{K}$, i.e., I is a periodically finitely generated p-ideal of R which is a contradiction. Thus for any a, $b \in R$, $ab \in I \Rightarrow a \in I$ or $b \in I$. So the p-ideal I is a p-prime ideal of R (since R is commutative).

Definition 4.4. A semiring R is said to be *p-noetherian* if it satisfies the ascending chain conditions on p-ideals.

This is clear from the definition that a noetherian semiring is always p-noetherian.

Following is an example of a p-noetherian semiring which is not noetherian.

Example 4.5. For two positive integers $m > 1$ and $n > 1$, we consider the commutative semiring $R = T(m, n)$ which is defined in exmple 4.1. Let $I_1 \subseteq I_2 \subseteq I_3 \subseteq \ldots$ be an ascending chain of p-ideals in R. If possible let the chain considered be non-terminating. Then for each $k \in \mathrm{N}$, we an choose an element $(\bar{x}_k, \alpha_k) \in I_k$ such that for $k > 1$, $(\bar{x}_k, \alpha_k) \notin I_{k-1}$. Now for each k and for any $\alpha \in C = [0, 1)$ with $\alpha \leq \alpha_k$, we see that $(\bar{x}_k, \alpha) = (\bar{1}, \alpha) \bullet (\bar{x}_k, \alpha_k) \in I_k$. Again for any $\alpha > \alpha_k$, $(\bar{x}_k, \alpha_k) + (\bar{x}_k, \alpha) = 2(\bar{x}_k, \alpha) \Rightarrow (\bar{x}_k, \alpha) \in \widehat{I_k} = I_k$. Thus for each $k \in \mathrm{N}$, $(\bar{x}_k, \alpha) \in I_k$, $\forall \alpha \in C$. Then, by the choice of the elements $(\bar{x}_k, \alpha_k)$ from I_k, we can say that the set $K = \{\bar{x}_k : k \in \mathrm{N}\}$ is infinite which is a contradiction, since $K \subseteq S(m, n)$ and $S(m, n)$ is finite. Hence the chain considered is terminating and thus $R = T(m, n)$ is a p-noetherian semiring.

For a digit i, we now consider a mapping $i : \mathrm{N} \to [0, 1) = C$ defined by $i(k) = \alpha$ (for any $k \in \mathrm{N}$), where α has i in each of the first k decimal places and 0 everywhere else. Clearly, $0(k) = 0$ for any $k \in \mathrm{N}$. We choose and fix a digit $i \neq 0$ and consider for each $p \in \mathrm{N}$, the subset $R_p = \{(\bar{0}, i(k)) \in R : k = 1, 2, \ldots, p\}$ of R. Suppose for each $p \in \mathrm{N}$, I_p be the ideal of R generated by R_p. Let $(\bar{x}, \alpha) \in I_p$ for some $p \in \mathrm{N}$. Then $(\bar{x}, \alpha) = \sum_{k=1}^{p} (\bar{y}_k, \beta_k) \bullet (\bar{0}, i(k))$. Clearly then $\bar{x} = \bar{0}$. Moreover, $\alpha = max.\{min.\{\beta_k, i(k)\}\}; k = 1, 2, \ldots, p\}$. Now for any k_1, $k_2 \in \mathrm{N}$ with $1 \leq k_1, k_2 \leq p$, we see that $i(k_1) \leq i(k_2)$ if and only if $k_1 \leq k_2$. So, $min.\{\beta_k, i(k)\} \leq i(k) \leq i(p)$ for all $k = 1, 2, \ldots, p$. Thus $\alpha \leq i(p)$. Hence $I_p \subseteq \{(\bar{0}, \alpha) : \alpha \leq i(p)\}$. Again for any $\alpha \in C$ with $\alpha \leq i(p)$, since $(\bar{0}, i(p)) \in R_p$ and $(\bar{0}, \alpha) = (\bar{x}, \alpha) \bullet (\bar{0}, i(p))$ for any $(\bar{x}, \alpha) \in R$, so $\{(\bar{0}, \alpha) : \alpha \leq i(p)\} \subseteq I_p$, whence $I_p = \{(\bar{0}, \alpha) : \alpha \leq i(p)\}$ for all $p \in \mathrm{N}$. Thus $I_1 \underset{\neq}{\subset} I_2 \underset{\neq}{\subset} I_3 \ldots$ is a non-terminating ascending chain of ideals in R. So, R is not noetherian.

Proposition 4.6. *The following conditions on a semiring R with absorbing zero 0_R are equivalent:*

(1) *R is p-noetherian;*

(2) *Any non-empty collection of p-ideals of R has a maximal element;*

(3) *Every p-ideal of R is periodically finitely generated.*

Proof. (1) $\Rightarrow$ (2) : Let c be a non-empty collection of *p*-deals of R. We choose $I_1 \in c$. If $I_1 \not\subseteq I$ for any $I \in c$, other than I_1, then I_1 is a maximal element of c. If not, then $I_1 \subsetneq I_2$, $I_1 \not\subseteq I$ for any $I \in c$, other than I_1, then I_1 is a maximal element of c. If not, then $I_1 \subsetneq I_2$, *for some* $I_2 \in c$. If $I_2 \not\subseteq I$ for any $I \in c$, other than I_2, then I_2 is a maximal element of c. If not, then $I_1 \subsetneq I_2 \subsetneq I_3$, for some $I_3 \in c$. Since R is *p*-noetherian and every element of c is a *p*-ideal, so this process must end after a finite number of steps. Thus c has a maximal element.

(2) $\Rightarrow$ (3) : Let I be a *p*-ideal of R and c be the collection of all periodically finitely generated *p*-ideals of R contained in I. Since, R has absorbing zero 0_R, so $P^+(R)$ is a periodically finitely generated *p*-ideal of R and also $P^+(R) \subseteq I$. Thus $P^+(R) \in c$, i.e., $c \neq \phi$. So, by (2), c has a maximal element H. Let $H = (\widehat{a_1, \ldots, a_m})$, for some $a_i \in R$, $(i = 1, 2, \ldots, m;\ m \in \mathbb{N})$. For each $b \in I$, let us consider the periodically finitely generated *p*-ideal $H_b = (\widehat{a_1, \ldots, a_m}, b)$. Clearly $H_b \in c$ and $H \subseteq H_b$. Hence by maximality of H in C, $H = H_b$. So, $b \in H$ for all $b \in I$. Thus $I = H$ and so I is periodically finitely generated.

(3) $\Rightarrow$ (1) : Let $I_1 \subseteq I_2 \ldots$ be an ascending chain of *p*-ideals of R. Suppose $I = \cup_{j=1}^{\infty} I_j$. Clearly I is a *p*-ideal of R. Hence by (3), I is periodically finitely generated. Suppose that $I = (\widehat{a_1, \ldots, a_m})$, for some $a_i \in R$, $(i = 1, 2, \ldots, m;\ m \in \mathbb{N})$. So, there exists $n \in \mathbb{N}$ such that $a_i \in I_n$, $\forall i$ and hence $I = (\widehat{a_1, \ldots, a_m}) \subseteq I_n$, since I_n is a *p*-ideal of R. Consequently, $I_n = I$ and so, $I_j = I_n$, $\forall j \geq n$. Thus, (1) follows.

Proposition 4.7. *Let R be a commutative semiring with absorbing zero such that each of its p-ideal is also a k-ideal. Then R is p-noetherian if and only if every p-prime ideal of R is peridocially finitely generated.*

Proof. If R is *p*-noetherian, then, by proposition 4.6, every *p*-prime ideal of R is priodically finitely generated.

Conversely, let every *p*-prime ideal of R is periodically finitely generated. Suppose, if possible, the class c of all those *p*-ideals of R which are not periodicaly finitely generated is not empty. Let $\{I_j : j \in \Lambda\}$ be a chain of elements of c. Then $I = \cup_{j=1}^{\infty} I_j$, is a *p*-ideal which is not periodically finitely generated. In fact, if $I = (\widehat{a_1, \ldots, a_m})$, for some $a_i \in R$, $(i = 1, 2, \ldots, m;\ m \in \mathbb{N})$, then there exists some I_j such that $a_i \in I_j$, $\forall i$ and then clearly we have $I = (\widehat{a_1, \ldots, a_m}) \subseteq I_j$ (since I_j is a *p*-ideal). Then $I_j = I$ which says that I_j is periodically finitely generated which is a contradiction, since $I_l \in c$ and c does not contain any periodically finitely generated p-ideal. Thus, by Zorn's lemma, c has a maximal element I_0. So, by proposition 4.3, I_0 is a p-prime ideal, contradicting the hypothesis that all p-prime ideals of R are periodically finitely generated. Thus our assumption that c is non-empty, is false. So every p-ideal of R is periodically finitely generated. Thus by proposition 4.5, R is p-noetherian.

5. p-PRIME RADICAL OF AN IDEAL

Suppose $Spec_p(R)$ denotes the set of all p-prime ideals of a semiring R. Let I be a proper ideal of the semiring R and $V_p(I) = \{H \in Spec_p(R) : I \subseteq H\}$. Then by proposition 3.3, $\cap\ V_p(I)$ is a p-semiprime ideal of R.

Definition 5.1. [2] For any proper ideal I of R, the p-semiprime ideal $\cap V_p(I)$ is called the *p-prime radical* (simply the *p-radical*) of I and is denoted by $\sqrt[p]{I}$ or $Rad_p(I)$.

Proposition 5.2. [2] *Let I be an ideal of a commutative semiring R. Then*

$$\sqrt[p]{I} = \{a \in R \mid a^m \in \hat{I} \text{ for some } m \in \mathrm{N}\}.$$

The notion of p-radical of an ideal in a semiring is studied elaborately in [2]. In continuation to that, we present the next proposition, towards which we first require the following lemma.

Lemma 5.3. *For any ideal J of a commutative semiring R, if $a \in \hat{J}$, then for every $n \in \mathrm{N}$, there exist $\alpha \in J$ and $l_1 \in \mathrm{N}$ such that*

$$\alpha^n + l_n a^n = (l_n + 1)a^n \ \ldots\ (*)$$

where for $n \geq 1$, $l_n = (l_1 + 1)^n - 1 \in \mathrm{N}$.

Proof. We shall prove the result by using strong induction over N. Since $a \in \hat{J}$, so there exist $\alpha \in J$ and $l_1 \in \mathrm{N}$ such that $\alpha + l_1 a = (l_1 + 1)a$. So, (*) is true for $n = 1$. Suppose (*) is true for $n = 1, 2, \ldots, m$ for some $m \geq 1$. Then, $\alpha^m + l_m a^m = (l_m + 1)a^m$

$\Rightarrow \alpha^{m+1} + l_m a^m \alpha = (l_m + 1)a^m \alpha$

$\Rightarrow \alpha^{m+1} + l_m a^m \alpha + l_1(l_m + 1)a^{m+1} = (l_m + 1)a^m \alpha + l_1(l_m + 1)a^{m+1}$

$\Rightarrow \alpha^{m+1} + l_m a^m(\alpha + l_1 a) + l_1 a^{m+1} = (l_m + 1)a^m(\alpha + l_1 a)$

$\Rightarrow \alpha^{m+1} + l_m(l_1 + 1)a^{m+1} + l_1 a^{m+1} = (l_m + 1)(l_1 + 1)a^{m+1}$

$\Rightarrow \alpha^{m+1} + ((l_m + 1)(l_1 + 1) - 1)a^{m+1} = (l_m + 1)(l_1 + 1)a^{m+1}$

$\Rightarrow \alpha^{m+1} + ((l_1 + 1)^m(l_1 + 1) - 1)a^{m+1} = (l_1 + 1)^m(l_1 + 1)a^{m+1}$

$\Rightarrow \alpha^{m+1} + ((l_1 + 1)^{m+1} - 1)a^{m+1} = (l_1 + 1)^{m+1}a^{m+1}$

$\Rightarrow \alpha^{m+1} + l_{m+1}a^{m+1} = (l_{m+1} + 1)a^{m+1}$.

Hence by strong induction (*) follows for all $n \in \mathrm{N}$.

Proposition 5.4. *Let I be an ideal of a commutative semiring R satisfying the condition that $\sqrt[p]{I}$ is periodically finitely generated. Then there exists a positive integer n such that $a^n \in \hat{I}$, for any $a \in \sqrt[p]{I}$.*

Proof. Suppose that $\sqrt[p]{I} = \hat{J}$ where $J = (a_1, \ldots, a_k)$ for some $a_i \in R$. For each $1 \leq i \leq k$, there exists $n_i \in \mathrm{N}$ such that $a_i^{n_i} \in \hat{I}$ (by proposition 5.2). Thus for each i, there exist $p_i \in \mathrm{N}$ and $x_i \in I$ such that $x_i + p_ia_i^{n_i} = (p_i + 1)a_i^{n_i}$. Suppose that $p = max.\{p_1, p_2, \ldots, p_k\}$, then we have

$$x_i + pa_i^{n_i} = (p+1)a_i^{n_i} \quad \ldots (1)$$

Now let $a \in \sqrt[p]{I} = \hat{J}$ be arbitrary. Suppose that $n = \sum_{i=1}^{k} n_i$. Then by lemma 5.3, there exist $r_i \in R$ and $l_1 \in \mathrm{N}$ such that

$$\left(\sum_{i=1}^{k} r_ia_i\right)^n + la^n = (l+1)a^n \quad \ldots (2),$$

where $l = l_n = (l_1 + 1)^n - 1$. Now since R is commutative, so

$$\left(\sum_{i=1}^{k} r_ia_i\right)^n = \sum \frac{n!}{h_1!\ldots h_k!}(r_1a_1)^{h_1}\ldots(r_ka_k)^{h_k} \quad \ldots (3),$$

where the sum is taken over all k-tuples $(h_1,\ldots,h_k)$ satisfying $\sum_{i=1}^{k} h_i = n$. We take m to denote the number of above mentioned k-tuples (as its exact expression in terms of n and k adds no merit to this proof). Let us fix a particular ordering of the summands in the sum of the right side of (3) and by that we consider the jth. summand $(1 \leq j \leq m)$ as $K_j = \frac{n!}{h_1!\ldots h_k!}(r_1a_1)^{h_1}\ldots$

$(r_k a_k)^{h_k}$. Since $\sum\limits_{i=1}^{k} h_i = n$, so there is $h_l \geq n_l$ for at least one index l. Then from (1), we have

$$x_l a_l^{h_l - n_l} + p a_l^{h_l} = (p+1) a_l^{h_l} \Rightarrow x_l a_l^{h_l - n_l} r_l^{h_l} + p a_l^{h_l} r_l^{h_l} = (p+1) a_l^{h_l} r_l^{h_l}$$

$$\Rightarrow x_l a_l^{h_l - n_l} r_l^{h_l} + p(a_l r_l)^{h_l} = (p+1)(a_l r_l)^{h_l}$$

$$\Rightarrow \left(x_l (a_1 r_1)^{h_1} \ldots \left(a_l^{h_l - n_l} r_l^{h_l}\right) \ldots (a_k r_k)^{h_k} \right) + p\left((a_1 r_1)^{h_1} \ldots (a_l r_l)^{h_l} \ldots (a_k r_k)^{h_k}\right)$$

$$= (p+1)\left((a_1 r_1)^{h_1} \ldots (a_l r_l)^{h_l} \ldots (a_k r_k)^{h_k}\right)$$

$$\Rightarrow \frac{n!}{h_1! \ldots h_k!}\left(x_l (a_1 r_1)^{h_1} \ldots \left(a_l^{h_l - n_l} r_l^{h_l}\right) \ldots (a_k r_k)^{h_k} \right) + pK_j = (p+1) K_j$$

$\Rightarrow$ there exists for each j, an element.

$$\alpha_j = \frac{n!}{h_1! \ldots h_k!}\left(x_l (a_1 r_1)^{h_1} \ldots \left(a_l^{h_l - n_l} r_l^{h_l}\right) \ldots (a_k r_k)^{h_k} \right) \in I$$

(since $x_i \in I$ for each i) such that $\alpha_j + pK_j = (p \div 1)K_j$

$$\Rightarrow \sum_{j=1}^{m} \alpha_j + p\left(\sum_{i=1}^{k} r_i a_i\right)^n = (p+1)\left(\sum_{i=1}^{k} r_i a_i\right)^n$$

$$\Rightarrow \alpha + p\left(\sum_{i=1}^{k} r_i a_i\right)^n + (p+1) l a^n = (p+1)\left(\sum_{i=1}^{k} r_i a_i\right)^n + (p+1) l a^n \quad \text{where } \alpha = \sum_{j=1}^{m} \alpha_j \in I.$$

$$\Rightarrow \alpha + p\left(\left(\sum_{i=1}^{k} r_i a_i\right)^n + l a^n\right) + l a^n = (p+1)\left(\left(\sum_{i=1}^{k} r_i a_i\right)^n + l a^n\right)$$

$$\Rightarrow \alpha + p(l+1)a^n + la^n = (p+1)(l+1)a^n \Rightarrow \alpha + (pl + p + l)a^n = (pl + p + l + 1)a^n$$

$$\Rightarrow a^n \in \hat{I}.$$

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REFERENCES

1. Dasgupta, Utpal and Mukhopadhyay, P. (2005) : On *p*-prime ideals in semirings; Southeast Asian Bull. Math., 29, 859-872.
2. Dasgupta, Utpal and Mukhopadhyay, P. : *p*-radical of a semiring; Southeast Asian Bull. Math. (to appear).
3. J. S. Golan (1992) : The theory of semirings with applications in mathematics and theoretical computer science, Pitman Monographs and Surveys in Pure and Applied Mathematics, 54, Longman Sci. Tech., Harlow.
4. Mukhopadhyay, P. and Ghosh, S. (1999) : A new class of ideals in semirings; Southeast Asian Bull. Math., 23, 253-264.
5. Sen, M. K. and Adhikari, M.R. (1993) : On maximal *k*-ideals of semirings; Proc. amer. Math. Soc., 118, 699-703.
6. Sen, M. K., Mukhopadhyay, P. and Ghosh, S. (1999) : A study of a new class of ideals in semirings; FILOMAT, 13, 41-52.
7. Sen, M. K., Mukhopadhyay, P. and Ghosh, S. (2002) : *p*-ideals in *p*-regular semirings; Southeast Asian Bull. Math., 26(4), 439-452.
8. Shabir, M. and Hussain, S. (2009) : Semiring charecterised by the properties of their *p*-ideals; Southeast Asian Bull. Math., 33, 349-359.

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A NEW VIEWPOINT IN THE STUDY OF g-CLOSED FUNCTIONS

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ABSTRACT : By using *gm*-closed sets [35], we define *gm*-closed functions parallel to *g*-closed functions and obtain the unified properties of some modifications of *g*-closed functions. We obtain some preservation theorems of modifications of regular spaces and normal spaces.

Key words and pharases : *m*-structure, *m*-open set, *gm*-closed function, *gM*-closed function, pre-*gM*-closed function, uasi *gm*-closed function, *m*-regular space, *m*-normal space.
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1. INTRODUCTION

The concept of generalized closed (*g*-closed) sets in a topological space was initiated by Levine [20] in 1970. These sets were also considered by Dunham [14] and Dunham and Levine [15]. Since then, many modifications of *g*-closed sets were introduced and investigated. among them, the following sets are well-known; α*g*-closed [10], *gs*-closed [9], *gp*-closed [4], *gsp*-closed [11], γ*g*-closed [17]. Recently, the first author [35] of the present paper introduced the notion of *gm*-closed sets and tried to unify the above modifications of *g*-closed sets. Munshi and Bassan [29] nd Malghan [23] introduced the notion of *g*-closed functions in topological spaces. Moreover, functions called *gs*-closed [8], *gp*-closed [36], α*g*-closed [33], γ*g*-closed [17], *gsp*-closed [17] are defined and investigated.

The present authors [41], [42] introduced the notions of *m*-structures, *m*-spaces and *m*-continuity. In [38], the authors defined and investigated *m*-closed functions between a topological space and an *m*-space. In this paper, by using *gm*-closed sets we define *gm*-closed functions parallel to g-closed functions and obtain the unified properties of the above modifications of g-closed functions. Moreover, we define and investigate g*M*-closed, quasi g*m*-closed and pre-g*M*-closed functions. In the last section, we obtain some preservation theorems of modifications of regular spaces and normal spaces.

2. PRELIMINARIES

Let (X, τ) be a topological space and A a subset of X. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$, respectively.

Definition 2.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) *Semi-open* [19] if $A \subset \mathrm{Cl}(\mathrm{Int}(A))$,

(2) *preopen* [24] if $A \subset \mathrm{Int}(\mathrm{Cl}(A))$;

(3) *α-open* [31] if $A \subset \mathrm{Int}(\mathrm{Cl}(\mathrm{Int}(A)))$,

(4) *γ-open* [17] if $A \subset \mathrm{Int}(\mathrm{Cl}(A)) \cup \mathrm{Cl}(\mathrm{Int}(A))$,

(5) *β-open* [1] or *semi-preopen* [3]) if $A \subset \mathrm{Cl}(\mathrm{Int}(\mathrm{Cl}(A)))$.

The family of all semi-open (resp. preopen, α-open, γ-open, β-open) sets in (X, τ) is denoted by $\mathrm{SO}(X)$ resp. $\mathrm{PO}(X)$, $\alpha(X)$, $\gamma(X)$, $\beta(X)$ or $\mathrm{SPO}(X)$). Among the above generalizations of open sets, the following implications are well-known:

DIAGRAM I

$$\begin{tikzcd}
\text{Open} \arrow[r, Rightarrow] & \alpha\text{-open} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{preopen} \arrow[d, Rightarrow] & \\
& \text{semi-open} \arrow[r, Rightarrow] & \gamma\text{-open} \arrow[r, Rightarrow] & \text{semi-preopen}
\end{tikzcd}$$

Definition 2.2. The complement of a semi-open (resp. preopen, α-open, γ-open, β-open) set is said to be *semi-closed* [7] (resp. *preclosed* [24], *α-closed* [25], *γ-closed* [17], *β-closed* [1] or *semi-preclosed* [3]).

Definition 2.3. The intersection of all semi-closed (resp. preclosed, α-closed, γ-closed, β-closed) sets of X containing A is called the *semi-closure* [7] (resp. *preclosure* [16], *α-closure* [25], *γ-closure* [17], *β-closure* [2] or *semi-preclosure* [3]) of A and is denoted by $\mathrm{sCl}(A)$ (resp. $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$).

Definition 2.4. The union of all semi-open (resp. preopen, α-open, γ-open, β-open) sets of X contained in A is called the *semi-interior* (resp. *preinterior*, *α-interior*, *γ-interior*, *β-interior* or *semi-preinterior*) of A and is denoted by $\mathrm{sInt}(A)$ (resp. $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\gamma\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

Definition 2.5. A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is said to be

(1) *semi-closed* [32] (resp. *preclosed* [16], *α-closed* [25], *γ-closed* [17], *β-closed* [1]) if $f(A)$ is semi-closed (resp. preclosed, α-closed, γ-closed, β-closed) for each closed set A of X,

(2) *presemiclosed* [18] (resp. *M*-preclosed [26], *strongly α-closed* [30], *pre-β-closed* [21]) if $f(A)$ is semi-closed (resp. preclosed, α-closed, β-closed) in Y for every semi-closed (resp. preclosed, α-closed, β-closed) set A of X,

(3) *quasi α-closed* [30] if $f(A)$ is closed in Y for every α-closed set A of X.

3. MINIMAL STRUCTURES AND *m*-CLOSED FUNCTIONS

Definition 3.1. a subfamily m_X of the power set $\mathcal{P}(X)$ of a nonempty set X is called a *minimal structure* (briefly *m-structure*) on X [41] if $\emptyset \in m_X$ and $X \in m_X$.

By (X, m_X), we denote a nonempty set X with a minimal structure m_X on X and call it an *m-space*. Each member of m_X is said to be *m_X-open* (or briefly *m-open*) and the complement of an m_X-open set is said to be *m_X-closed* (or briefly *m-closed*).

Remark 3.1. Let (X, τ) be a topological space. Then the families τ, SO(X), PO(X), $\alpha(X)$, $\gamma(X)$, and $\beta(X)$ are all m-structures on X.

Definition 3.2. Let (X, m_X) be an m-space. For a subset A of X, the *m-closure* of A and the *m-interior* of A are defined in [22] as follows:

(1) $\mathrm{mCl}(A) = \cap\{F : A \subset F, X \setminus F \in m_X\}$,

(2) $\mathrm{mInt}(A) = \cup\{U : U \subset A, U \in m_X\}$.

Remark 3.2. Let (X, τ) be a topological space and A a subset of X. If $m_X = \tau$ (resp. SO(X), PO(X), $\alpha(X)$, $\gamma(X)$, $\beta(X)$), then we have

(1) $\mathrm{mCl}(A) = \mathrm{Cl}(A)$ (resp. $\mathrm{sCl}(A)$, $\mathrm{pCl}(A)$, $\alpha\mathrm{Cl}(A)$, $\gamma\mathrm{Cl}(A)$, $\beta\mathrm{Cl}(A)$ or $\mathrm{spCl}(A)$),

(2) $\mathrm{mInt}(A) = \mathrm{Int}(A)$ (resp. $\mathrm{sInt}(A)$, $\mathrm{pInt}(A)$, $\alpha\mathrm{Int}(A)$, $\gamma\mathrm{Int}(A)$, $\beta\mathrm{Int}(A)$ or $\mathrm{spInt}(A)$).

Lemma 3.1. (Maki et al. [22]) *Let X be a nonempty set and m_X a minimal structue on X. For subsets A and B of X, the following properties hold:*

(1) $\mathrm{mCl}(X \setminus A) = X \setminus \mathrm{mInt}(A)$ *and* $\mathrm{mInt}(X \setminus A) = X \setminus \mathrm{mCl}(A)$,

(2) *If* $(X \setminus A) \in m_X$, *then* $\mathrm{mCl}(A) = A$ *and if* $A \in m_X$, *then* $\mathrm{mInt}(A) = A$,

(3) $\mathrm{mCl}(\emptyset) = \emptyset$, $\mathrm{mCl}(X) = X$, $\mathrm{mInt}(\emptyset) = \emptyset$ *and* $\mathrm{mInt}(X) = X$,

(4) *If* $A \subset B$, *then* $\mathrm{mCl}(A) \subset \mathrm{mCl}(B)$ *and* $\mathrm{mInt}(A) \subset \mathrm{mInt}(B)$,

(5) $A \subset \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(A) \subset A$,

(6) $\mathrm{mCl}(\mathrm{mCl}(A)) = \mathrm{mCl}(A)$ *and* $\mathrm{mInt}(\mathrm{mInt}(A)) = \mathrm{mInt}(A)$.

Definition 3.3. A minimal structure m_X on a nonempty set X is said to have *property* $\mathcal{B}$ [22] if the union of any family of subsets belonging to m_X belongs to m_X.

Remark 3.3. Let (X, τ) be a topological space. Then SO(X), PO(X), $\alpha(X)$, $\gamma(X)$ and $\beta(X)$ have property $\mathcal{B}$.

Lemma 3.2. (Popa and Noiri [42]) *Let (X, m_X) an m-space, where m_X satisfies property $\mathcal{B}$. For a subset A of X, the following properties hold:*

(1) $A \in m_X$ *if and only if* $\mathrm{mInt}(A) = A$,

(2) A *is m_X-closed if and only if* $\mathrm{mCl}(A) = A$,

(3) $\mathrm{mInt}(A) \in m_X$ *and* $\mathrm{mCl}(A)$ *is m_X-closed.*

Definition 3.4. (1) A function $f : (X, \tau) \rightarrow (Y, m_Y)$ is said to be *m-closed* [38] if for each closed set F of (X, τ), $f(F)$ is m-closed in (Y, m_Y).

(2) A function $f : (X, m_X) \rightarrow (Y, m_Y)$ is said to be *M-closed* [38] if for each m-closed set F of (X, m_X), $f(F)$ is m-closed in (Y, m_Y).

(3) A function $f : (X, m_X) \rightarrow (Y, \sigma)$ is said to be *quasi m-closed* [38] if for each m-closed set F of (X, m_X), $f(F)$ is closed in (Y, σ).

Theorem 3.1. (Noiri and Popa [38]) *For a function $f : (X, \tau) \rightarrow (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is m-closed;*

(2) *for each subset F of Y and each open set U of X with $f^{-1}(F) \subset U$, there exists $V \in m_Y$ such that $F \subset V$ and $f^{-1}(V) \subset U$;*

(3) *for each $y \in Y$ and each open set U of X such that $f^{-1}(y) \subset U$, there exists $V \in m_Y$ containing y such that $f^{-1}(V) \subset U$;*

(4) $\mathrm{mCl}(f(A)) \subset f(\mathrm{Cl}(A))$ *for every subset A of X.*

Theorem 3.2. (Noiri and Popa [38]) *For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is M-closed;*

(2) *for each subset F of Y and each $U \in m_X$ with $f^{-1}(F) \subset U$, there exists $V \in m_Y$ such that $F \subset V$ and $f^{-1}(V) \subset U$;*

(3) *for each $y \in Y$ and each $U \in m_X$ with $f^{-1}(y) \subset U$, there exists $V \in m_Y$ containing y such that $f^{-1}(V) \subset U$.*

Theorem 3.3. (Noiri and Popa [38]) *A function $f : (X, m_X) \rightarrow (Y, m_Y)$, where m_X and m_Y have property $\mathcal{B}$, is M-closed if and only if* $\mathrm{mCl}(f(A)) \subset f(\mathrm{mCl}(A))$ *for every subset A of X.*

Theorem 3.4. (Noiri and Popa [38]) *For a function $f : (X, m_X) \rightarrow (Y, m_Y)$, where m_Y has property $\mathcal{B}$, the following properties are equivalent:*

(1) *f is M-closed;*

(2) *if $U \in m_X$, then the set $\{y \in Y : f^{-1}(y) \subset U\}$* is *$m_Y$-open in Y;*

(3) *if F is m_X-closed, then the set $\{y \in Y : f^{-1}(y) \cap F \neq \}$ is m_Y-closed in Y.*

Theorem 3.5. (Noiri and Popa [38]) *For a function $f : (X, m_X) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is quasi m-closed;*

(2) *for ech subset F of Y and each $U \in m_X$ with $f^{-1}(F) \subset U$, there exists an open set V of Y such that $F \subset V$ and $f^{-1}(V) \subset U$;*

(3) *for each $y \in Y$ and each $U \in m_X$ with $f^{-1}(y) \subset U$, there exists an open set V of Y containing y such that $f^{-1}(V) \subset U$.*

Theorem 3.6. (Noiri and Popa [38] *A function $f : (X, m_X) \to (Y, \alpha)$, where m_X has property $\mathcal{B}$, is quasi m-closed if and only if* $\mathrm{Cl}(f(A)) \subset f(\mathrm{mCl}(A))$ *for evry subset A of X.*

4. *gm*-CLOSED FUNCTIONS

Definition 4.1. Let (X, τ) be a topological space. A subset A of X is said to be

(1) g-*closed* [20] if $\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,

(2) αg-*closed* [10] if $\alpha\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,

(3) g*s-closed* [9] if $\mathrm{sCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,

(4) g*p-closed* [4] if $\mathrm{pCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,

(5) g*sp-closed* [11] if $\mathrm{spCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$,

(6) γg-*closed* [17] if $\gamma\mathrm{Cl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

Definition 4.2. A subset A of a topological space (X, τ) is said to be g-*open* (resp. g*s-open*, g*p-open*, αg-*open*, γg-*open*, g*sp-open*) if the complement of A is g-closed (resp. g*s*-closed, g*p*-closed, αg-closed, γg-closed, g*sp*-closed).

The family of all g-open (resp. g*s*-open, g*p*-open, αg-open, γg-open, g*sp*-open) sets of a topological space (X, τ) is denoted by GO(X) (resp. GSO(X), GPO(X), αGO(X), γGO(X), GSPO(X)).

Definition 4.3. Let (X, τ) be a topologica space and A be a subset of X. The intersection of all g-closed (resp. g*s*-closed, g*p*-closed, αg-closed, γg-closed, g*sp*-closed) sets of X containing

A is called the g-*closure* [14] (resp. gs-*closure*, gp-*closure*, αg-*closure*, γg-*closure*, gsp-*closure*) of A and is denoted by $\mathrm{Cl}_g(A)$ (resp. $\mathrm{sCl}_g(A)$, $\mathrm{pCl}_g(A)$, $\alpha\mathrm{Cl}_g(A)$, $\gamma\mathrm{Cl}_g(A)$, $\mathrm{spCl}_g(A)$).

Definition 4.4. Let (X, τ) be a topologica space and A be a subset of X. the union of all g-open (resp. gs-open, gp-open, αg-open, γg-open, gsp-open) sets of X contained in A is called the g-*interior* [6] (resp. gs-*interior*, gp-*interior*, αg-*interior*, γg-*interior*, gsp-*interior*) of A and is denoted by $\mathrm{Int}_g(A)$ [6] (resp. $\mathrm{sInt}_g(A)$, $\mathrm{pInt}_g(A)$, $\alpha\mathrm{Int}_g(A)$, $\gamma\mathrm{Int}_g(A)$, $\mathrm{spInt}_g(A)$).

Remark 4.1. Let (X, τ) be a topological space and A be a subset of X.

(1) $\mathrm{GO}(X)$, $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\gamma\mathrm{GO}(X)$ and $\mathrm{GSPO}(X)$ are all m-structures on X. Hence if we put $m_X = \mathrm{GO}(X)$ (resp. $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\gamma\mathrm{GO}(X)$, $\mathrm{GSPO}(X)$), then we have

(i) $\mathrm{mCl}(A) = \mathrm{Cl}_g(A)$ (resp. $\mathrm{sCl}_g(A)$, $\mathrm{pCl}_g(A)$, $\alpha\mathrm{Cl}_g(A)$, $\gamma\mathrm{Cl}_g(A)$, $\mathrm{spCl}_g(A)$),

(ii) $\mathrm{mInt}(A) = \mathrm{Int}_g(A)$ (resp. $\mathrm{sInt}_g(A)$, $\mathrm{pInt}_g(A)$, $\alpha\mathrm{Int}_g(A)$, $\gamma\mathrm{Int}_g(A)$, $\mathrm{spInt}_g(A)$).

(2) If $m_X = \mathrm{GO}(X)$, then by Lemma 3.1 we obtain the results established in Theorem 2.1 (4), (5) and Theorem 2.2 (2), (3), (5), (6) of [6].

(3) The m-structures $\mathrm{GO}(X)$, $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\gamma\mathrm{GO}(X)$ and $\mathrm{GSPO}(X)$ do nto satisfy property $\mathcal{B}$, in general.

Definition 4.5. Let (X, τ) be a topological space and m_X a minimal structure on X. A subset A is said to be *generalized m-closed* (briefly gm-*closed*) [35] if $\mathrm{mCl}(A) \subset U$ whenever $A \subset U$ and $U \in \tau$.

The complement of a gm-closed set is said to be gm-*open*. The family of all gm-open sets is denoted by $\mathrm{GMO}(X)$. Obviously, $\mathrm{GMO}(X)$ is an m-structure on X and is called a gm-*structure* on X.

Remark 4.2. Let (X, τ) be a topological space and m_X an m-structure on X. We put $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\alpha(X)$, $\alpha(X)$, $\beta(X)$). Then a gm-closed set is g-closed (resp. gs-closed, gp-closed, αg-closed, γg-closed, gsp-closed).

By Remrk 3.2 of [17], the following diagram holds:

DIAGRAM II

$$\begin{tikzcd}
\text{open} \arrow[r, Rightarrow] & \text{g-open} \arrow[r, Rightarrow] & \alpha\text{g-open} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{gp-open} \arrow[d, Rightarrow] & \\
 & & \text{gs-open} \arrow[r, Rightarrow] & \gamma\text{g-open} \arrow[r, Rightarrow] & \text{g}sp\text{-open}
\end{tikzcd}$$

Definition 4.6. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *g-closed* [23] (resp. *gs-closed* [8] *gp-closed* [36], *δg-closed* [33], *γg-closed* [17], *gsp-closed* [17] if $f(F)$ is g-closed (resp. gs-closed, gp-closed, αg-closed, γg-closed, gsp-closed) for each closed set F of X.

Definition 4.7. Let (X, τ) be a topological sapce and GMO(Y) a gm-structure on (Y, σ). A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *gm-closed* if $f : (X, \tau) \to (Y, \mathrm{GMO}(Y))$ is m-closed.

Remark 4.3. (1) If GMO(Y) is GO(Y) (resp. GSO(Y), GPO(Y), αGO(Y), γGO(Y), GSPO(Y)) and a function $f : (X, \tau) \to (Y, \sigma)$ is gm-closed, then we obtain Definition 4.6.

(2) If $\tau \subset m_X$, then every closed function is gm-closed. The converse is not true by Remark 5.2 of [17].

Lemma 4.1. (Noiri and Popa [38]) *Let* m_Y^1 *and* m_Y^2 *be two m-structures on a nonempty set Y. If* $m_Y^1 \subset m_Y^2$ *and a function* $f : (X, m_X) \to (Y, m_Y^1)$ *is M-closed, then* $f : (X, m_X) \to (Y, m_Y^2)$ *is M-closed.*

Corollary 4.1. *Let* (X, τ) *and* (Y, σ) *be topological spaces. If* $\mathrm{GM}^1\mathrm{O}(Y) \subset \mathrm{GM}^2\mathrm{O}(Y)$ *and* $f : (X, \tau) \to (Y, \sigma)$ *is* gm^1*-closed, then* f *is* gm^2*-closed.*

By Corollary 4.1, for a function $f : (X, \tau) \to (Y, \sigma)$ we obtain the following Diagram established in Remark 5.2 of [17]:

DIAGRAM III

$$\begin{tikzcd}
\text{closed} \arrow[r, Rightarrow] & \text{g-closed} \arrow[r, Rightarrow] & \alpha\text{g-closed} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{gp-closed} \arrow[d, Rightarrow] & \\
 & & \text{gs-closed} \arrow[r, Rightarrow] & \gamma\text{g-closed} \arrow[r, Rightarrow] & gsp\text{-closed}
\end{tikzcd}$$

Definition 4.8. Let (X, τ) be a topological space and GMO(X) a gm-structure on X. For a subset A of X, the *gm-closure* of A and the *gm-interior* of A are defined as follows:

(1) $\mathrm{mCl_g}(A) = \cap\{F : A \subset F, X \setminus F \in \mathrm{GMO}(X)\}$,

(2) $\mathrm{mInt_g}(A) = \cup\{U : U \subset A, U \in \mathrm{GMO}(X)\}$.

By Definition 4.7 and Theorem 3.1, we obtain the following theorem.

Theorem 4.1. *Let* (X, τ) *and* (Y, σ) *be topological spaces and* GMO(Y) *a gm-structure on Y. For a function* $f : (X, \tau) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) *f is gm-closed;*

(2) *for each subset F of Y and each open set U of X with* $f^{-1}(F) \subset U$, *there exists* $V \in \mathrm{GMO}(Y)$ *such that* $F \subset V$ *and* $f^{-1}(V) \subset U$;

(3) *for each* $y \in Y$ *and each open set* U *of* X *such that* $f^{-1}(y) \subset U$, *there exists* $V \in$ $\mathrm{GMO}(Y)$ *containing* y *such that* $f^{-1}(V) \subset U$;

(4) $\mathrm{mCl}_g(f(A)) \subset f(\mathrm{Cl}(A))$ *for every subset* A *of* X.

Remark 4.4. If $\mathrm{GMO}(Y) = \mathrm{GO}(Y)$ (resp. $\mathrm{GSO}(Y)$, $\mathrm{GPO}(Y)$, $\alpha\mathrm{GO}(Y)$), then by The-orem 4.1 we obtain the results established in Theorem 1.13 of [23] and Theorem 1 of [5] (resp. Theorem 4.5 of [8], Theorem 1 of [36], Theorem 5.5 of [17]).

Definition 4.9. Let (X, τ) and (Y, σ) be topological spaces and $\mathrm{GMO}(X)$ (resp. $\mathrm{GMO}(Y)$) be a *gm*-structure on X (resp. Y). A function $f : X, \tau) \to (Y, \sigma)$ is said to be *gM-closed* in the sense of Munshi and Bassan [29] (resp. *gsM-closed, gpM-closed, αgM-closed, γgM-closed, gspM-closed*) if $f(F)$ is *g*-closed (resp. *gs*-closed, *gp*-closed, αg-closed, γg-closed, *gsp*-closed) for each g-closed (resp. *gs*-closed, *gp*-closed, αg-closed, γg-closed, *gsp*-closed) set F of X.

Definition 4.10. Let (X, τ) and (Y, σ) be topologial spaces and $\mathrm{GMO}(X)$ (resp. $\mathrm{GMO}(Y)$) be a *gm*-structure on X (resp. Y). A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *gM-closed* if $f :$ $(X, \mathrm{GMO}(X)) \to (Y, \mathrm{GMO}(Y))$ is *M*-closed.

Remark 4.5. If $\mathrm{GMO}(X)$ is $\mathrm{GO}(X)$ (resp. $\mathrm{GSO}(X)$, $\mathrm{GPO}(X)$, $\alpha\mathrm{GO}(X)$, $\gamma\mathrm{GO}(X)$, $\mathrm{GSPO}(X)$), $\mathrm{GMO}(Y)$ is $\mathrm{GO}(Y)$ (resp. $\mathrm{GSO}(Y)$, $\mathrm{GPO}(Y)$, $\alpha\mathrm{GO}(Y)$, $\gamma\mathrm{GO}(Y)$, $\mathrm{GSPO}(Y)$) and a function $f :$ $(X, \tau) \to (Y, \sigma)$ is g*M*-closed, then by Definition 4.10 we obtain Definition 4.9.

Corollary 4.2. *Let* (X, τ) *and* (Y, σ) *be topological spaces,* $\mathrm{GMO}(X)$ *a gm-structure on* X *and* $\mathrm{GM^1O}(Y)$ *and* $\mathrm{GM^2O}(Y)$ *gm-structures on* Y. *If* $\mathrm{GM^1O}(Y) \subset \mathrm{GM^2O}(Y)$ *and* $f : (X, \tau) \to (Y,$ $\sigma)$ *is* gM^1*-closed, then* f *is* gM^2*-closed.*

Proof. This follows from Lemma 4.1.

By Corollary 4.2, for a function $f : (X, \tau) \to (Y, \sigma)$ we obtain the following Diagram:

DIAGRAM IV

$$\begin{tikzcd}
g\mathit{M}\text{-closed} \arrow[r, Rightarrow] & \alpha g\mathit{M}\text{-closed} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & gp\mathit{M}\text{-closed} \arrow[d, Rightarrow] & \\
& gs\mathit{M}\text{-closed} \arrow[r, Rightarrow] & \gamma g\mathit{M}\text{-closed} \arrow[r, Rightarrow] & gsp\mathit{M}\text{-closed}
\end{tikzcd}$$

By Definition 4.10, Theorems 3.2, 3.3 and 3.4, we obtain the following theorems:

Theorem 4.2. *Let* (X, τ) *and* (Y, σ) *be topological spaces and* $\mathrm{GMO}(X)$ (*resp.* $\mathrm{GMO}(Y)$) *be a gm-structure on* X (*resp.* Y), *where* $\mathrm{GMO}(Y)$ *has property* $\mathcal{B}$. *For a function* $f : (X, \tau) \to$ (Y, σ), *the following properties are equivalent:*

(1) *f is gM-closed;*

(2) *for each subset F of Y and each $U \in$* GMO(X) *with $f^{-1}(F) \subset U$, there exists $V \in$* GMO(Y) *such that $F \subset V$ and $f^{-1}(V) \subset U$;*

(3) *for each $y \in Y$ and each $U \in$* GMO(X) *with $f^{-1}(y) \subset U$, there exists $V \in$* GMO(Y) *containing y such that $f^{-1}(V) \subset U$.*

Theorem 4.3. *Let (X, τ) and (Y, σ) be topological spaces,* GMO(X) *(resp.* GMO(Y)) *be a gm-structure on X (resp. Y), where* GMO(X) *and* GMO(Y) *have property $\mathcal{B}$. Then a function f: $(X, \tau) \to (Y, \sigma)$ is gM-closed if and only if* $\mathrm{mCl}_g(f(A)) \subset f(\mathrm{mCl}_g(A))$ *for every subset A of X.*

Theorem 4.4. *Let (X, τ) and (Y, σ) be topological spaces and* GMO(X) *(resp.* GMO(Y)) *be a gm-structure on X (resp. Y), where* GMO(Y) *has property $\mathcal{B}$. Then for a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is gM-closed;*

(2) *if $U \in$* GMO(X), *then the set $\{y \in Y : f^{-1}(y) \subset U\}$ is gm-open in Y;*

(3) *if F is gm-closed in X, then the set $\{y \in Y : f^{-1}(y) \cap F \neq \emptyset\}$ is gm-closed in Y.*

Definition 4.11. Let (X, τ) and (Y, σ) be topological spaces, m_X an m-structure on X and GMO(Y) be a gm-structure on Y. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *pre-g-closed* (resp. *pre-gs-closed, pre-gp-closed* [36], *pre-αg-closed, pre-γg-closed, pre-gsp-closed)* if $f(F)$ is g-closed (resp. semi-closed, preclosed, α-closed, γ-closed, β-closed) set F of X.

Definition 4.12. Let (X, τ) and (Y, σ) be topological spaces, m_X an m-structure on X and GMO(Y) a gm-structure on Y. A funtion $f : (X, \tau) \to (Y, \sigma)$ is said to be *pre-gM-closed* if $f : (X, m_X) \to (Y,$ GMO(Y)) is M-closed.

Remark 4.6. Let (X, τ) and (Y, σ) be topological spaces, m_X an m-structure on X and GMO(Y) a gm-structure on Y. If m_X is τ (resp. SO(X), PO(X), $\alpha(X)$, $\gamma(X)$, $\beta(X)$), GMO(Y) is GO(Y) (resp. GSO(Y), GPO(Y), αGO(Y), γGO(Y), GSPO(Y)) and a function $f : (X, \tau) \to (Y, \sigma)$ is *pre-gM*-closed, then by Definition 4.12 we obtain Definition 4.11.

Lemma 4.2. (Noiri and Popa [38]) *Let m_X^1 and m_X^2 be two m-structures on a nnempty set X. If $m_X^1 \subset m_X^2$ and a function $f : (X, m_X^2) \to (Y, m_Y)$ is M-closed, then $f : (X, m_X^1) \to (Y, m_Y)$ is M-closed.*

Corollary 4.3. *Let (X, τ) and (Y, σ) be topological spaces, m_X^1, m_X^2 be two m-structures on*

X and GMO(Y) *a* g*m-structure on Y. If $m^1_X \subset m^2_X$ and a function $f : (X, \tau) \to (Y, \sigma)$ is pre-*g*M^2-closed, then f is pre-*g*M^1-closed.*

By Corollary 4.3., for a function $f : (X, \tau) \to (Y, \sigma)$ we obtain the following Diagram:

DIAGRAM V

$$\begin{tikzcd}
pre\text{-}gsp\text{-closed} \arrow[r, Rightarrow] & pre\text{-}\gamma g\text{-closed} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & pre\text{-}gs\text{-closed} \arrow[d, Rightarrow] & \\
 & pre\text{-}gp\text{-closed} \arrow[r, Rightarrow] & pre\text{-}\alpha g\text{-closed} \arrow[r, Rightarrow] & pre\text{-}g\text{-closed}
\end{tikzcd}$$

By Definition 4.12, Theorems 3.2 and 3.3, we obtain the following theorems:

Theorem 4.5. *Let (X, τ) and (Y, σ) be topological spaces, m_X an m-structure on X and* GMO(Y) *a δm-structure on Y with property $\mathcal{B}$. Then for a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is pre-gM-closed.*

(2) *for each subset F of Y and each $U \in m_X$ with $f^{-1}(F) \subset U$, there exists $V \in$* GMO(Y) *such that $F \subset V$ and $f^{-1}(V) \subset U$;*

(3) *for each $y \in Y$ and each $U \in m_X$ with $f^{-1}(y) \subset U$, there exists $V \in$* GMO(Y) *containing y such that $f^{-1}(V) \subset U$.*

Remark 4.7. If m_X = PO(X) and GMO(Y) = GPO(Y), then by Theorem 4.5 we obtain the result established in Theorem 1 of [36].

Theorem 4.6. *Let (X, τ) and (Y, σ) be topological spaces, m_X an m-structure on X with property $\mathcal{B}$ and* GMO(Y) *a δm-structure on Y with property $\mathcal{B}$. Then a function $f : (X, \tau) \to (Y, \sigma)$ is pre-gM-closed if and only if* $\mathrm{mCl}_g(f(A)) \subset f(\mathrm{mCl}(A))$ *for every subset A of X.*

Theorem 4.7. *Let (X, τ) and (Y, σ) be topological spaces, m_X an m-structur on X and* GMO(Y) *a gm-structure on Y with proerty $\mathcal{B}$. Then for a function $f : (X, \tau) \to (Y, \sigma)$, the following properties are equivalent:*

(1) *f is pre-*g*M-closed;*

(2) *if $U \in m_X$, then the set $\{y \in Y : f^{-1}(y) \subset U\}$ is gm-open in Y;*

(3) *if F is m-closed in X, then the set $\{y \in Y : f^{-1}(y) \cap F \neq \emptyset\}$ is* g*m-closed in Y.*

Definition 4.13. Let (X, τ) and (Y, σ) be topological spaces and GMO(X) a g*m*-structure on X. a function $f : (X, \tau) \to (Y, \sigma)$ is said to be *quasi* g*m-closed* if $f : (X,$ GMO(X))

$\to (Y, \sigma)$ is quasi m-closed, equivalently if $f(F)$ is closed in Y for each gm-closed set F of X.

Remark 4.8. Let (X, τ) be a topological spaces. If GMO(X) = GO(X) (resp. GSO(X), GPO(X), αGO(X), γGO(X), GSPO(X)) and $f : (X, \tau) \to (Y, \sigma)$ is quasi gm-closed; then f is quasi g-closed (resp. quasi gs-closed, quasi gp-closed, quasi αg-closed, quasi γg-closed, quasi gsp-closed).

Corollary 4.4. *Let* (X, τ) *and* (Y, σ) *be topological spaces,* $\mathrm{GM^1O}(X)$, $\mathrm{GM^2O}(X)$ *be two gm-structures on X. If* $\mathrm{GM^1O}(X) \subset \mathrm{GM^2O}(X)$ *and a function* $f : (X, \tau) \to (Y, \sigma)$ *is quasi* gM^2*-closed, then f is quasi* gM^1*-closed.*

Proof. The proof follows from lemma 4.2.

By corollary 4.4, for a function $f : (X, \tau) \to (Y, \sigma)$ we obtain the following Diagram:

DIAGRAM VI

$$\begin{tikzcd}
\text{quasi } gsp\text{-closed} \arrow[r, Rightarrow] & \text{quasi } \gamma g\text{-closed} \arrow[r, Rightarrow] \arrow[d, Rightarrow] & \text{quasi } gs\text{-closed} \arrow[d, Rightarrow] & \\
& \text{quasi } gp\text{-closed} \arrow[r, Rightarrow] & \text{quasi } \alpha g\text{-closed} \arrow[r, Rightarrow] & \text{quasi g-closed}
\end{tikzcd}$$

By Theorems 3.3, 3.4 and 3.5, we obtain the following theorems :

Theorem 4.8. *Let* (X, τ) *and* (Y, σ) *be topological spaces and* GMO(X) *be a gm-structure on X. For a function* $f : (X, \tau) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) *f is quasi gm-closed;*

(2) *for each subset F of Y and each* $U \in$ GMO(X) *with* $f^{-1}(F) \in U$, *there exists an open set V of Y such that* $F \subset V$ *and* $f^{-1}(V) \subset U$;

(3) *for each* $y \in Y$ *and each* $U \in$ GMO(X) *with* $f^{-1}(y) \subset U$, *there exists an open set V of Y containing y such that* $f^{-1}(V) \subset U$.

Theorem 4.9. *Let* (X, τ) *and* (Y, σ) *be topoogical spaces and* GMO(X) *be a gm-structure on X with property* $\mathcal{B}$. *Then a funtion* $f : (X, \tau) \to (Y, \sigma)$ *is quasi gm-closed if and only if* $\mathrm{Cl}(f(A)) \subset f(\mathrm{mCl_g}(A))$ *for every subset A of X.*

Theorem 4.10. *Let* (X, τ) *and* (Y, σ) *be topological spaces and* GMO(X) *a gm-structure on X with property* $\mathcal{B}$. *Then for a function* $f : (X, \tau) \to (Y, \sigma)$, *the following properties are equivalent:*

(1) *f is quasi gm-closed;*

(2) *if* $U \in \mathrm{GMO}(X)$, *then the set* $\{y \in Y : f^{-1}(y) \subset U\}$ *is open in* Y;

(3) *if* F *is gm-closed in* X, *then the set* $\{y \in Y : f^{-1}(y) \cap F \neq \emptyset\}$ *is closed in* Y.

5. PRESERVATIONS OF *m*-REGULARITY AND *m*-NORMALITY

Theorem 5.1. *Let* (X, τ) *and* (Y, σ) *be topological spaces,* m_X *an m-structure on* X *with property* $\mathcal{B}$ *and* $\mathrm{GMO}(Y)$ *a gm-structure on* Y. *If a function* $f : (X, \tau) \to (Y, \sigma)$ *is continuous pre-gM-closed and* A *is gm-closed in* X, *then* $f(A)$ *is gm-closed in* Y.

Proof. Let V be any open set of Y containing $f(A)$. Then $A \subset f^{-1}(V)$ and $f^{-1}(V)$ is open in X because f is continuous. Since A is *gm*-closed in X, $\mathrm{mCl}(A) \subset f^{-1}(V)$ and hence $f(A) \subset f(\mathrm{mCl}(A)) \subset V$. Since m_X has property $\mathcal{B}$, by Lemma 3.2 $\mathrm{mCl}(A)$ is *m*-closed in X. Since f is pre *gM*-closed, $f(\mathrm{mCl}(A))$ is *gm*-closed in Y and hence $\mathrm{mCl}(f(A)) \subset \mathrm{mCl}(f(m\mathrm{Cl}(A))) \subset V$. This shows that $f(A)$ is *gm*-closed in Y.

Remark 5.1. If $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$) and $\mathrm{GMO}(Y) = \mathrm{GO}(Y)$ (resp. $\mathrm{GSO}(Y)$, $\mathrm{GPO}(Y)$), then by Theorem 5.1 we obtain the result of Theorem 1.4 of [23] resp. Theorem 4.6 (ii) (iii) of [8], Proposition 3 of [36]).

Definition 5.1. An *m*-space (X, m_X) is said to be *m-regular* [43] if for each *m*-closed set F and each $x \notin F$, there exist disjoint *m*-open sets U and V such that $x \in U$ and $F \subset V$.

Remark 5.2. Let (X, τ) be a topological space and $m_X = \tau$ (resp. $\mathrm{SO}(X)$, $\mathrm{PO}(X)$, $\beta(X)$). Then *m*-regularity coincides with regularity (resp. semi-regularity [12], pre-regularity [40], semi-preregularity [34]).

Definition 5.2. Let (X, τ) be a topological spaces and $\mathrm{GMO}(X)$ a *gm*-structure on X. Then (X, τ) is said to be *gm-regular* if $(X, \mathrm{GMO}(X))$ is *m*-regular.

Remark 5.3. Let (X, τ) be a topological spaces. If $\mathrm{GMO}(X) = \mathrm{GO}(X)$, we obtain the definition of a *G*-regular space [28].

Definition 5.3. A function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-continuous* [41] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists $U \in m_X$ containing x such that $f(U) \subset V$.

Definition 5.4. Let (X, τ) and (Y, σ) be topological spaces and $\mathrm{GMO}(X)$ (resp. $\mathrm{GMO}(Y)$) a *gm*-structure on X (resp Y). A function $f : (X, \tau) \to (Y, \sigma)$ is said to be *gM-continuous* if $f : (X, \mathrm{GMO}(X)) \to (Y, \mathrm{GMO}(Y))$ is *M*-continuous.

Definition 5.5. a function $f : (X, m_X) \to (Y, m_Y)$ is said to be *M-open* [27] if for each $U \in m_X$, $f(U) \in m_Y$.

Definition 5.6. Let (X, τ) and (Y, σ) be topological spaces and GMO(X) (resp. GMO(Y)) a *gm*-structure on X (resp Y). A function $f : (X, \tau) \to (Y, \sigma)$ is said to be g*M-open* if $f : (X, \mathrm{GMO}(X)) \to (Y, \mathrm{GMO}(Y))$ is *M*-open.

Lemma 5.1. (Noiri and Popa [38]) *If $f : (X, m_X) \to (Y, m_Y)$, where m_X and m_Y have property $\mathcal{B}$, is an M-continuous, M-open and M-closed surjection and (X, m_X) is m-regular, then (Y, m_Y) is m-regular.*

Theorem 5.2. *Let (X, τ) and (Y, σ) be topological spaces and* GMO(X) *(resp.* GMO(Y)) *a gm-structure on X (resp. Y) having property B. If $f : (X, \tau) \to (Y, \sigma)$ is a* g*M-continuous,* g*M-open and* g*M-closed surjection and (X, τ) is gm-regular, then (Y, σ) is gm-regular.*

Proof. The proof follows from Lemma 5.1.

Remark 5.4. If $\mathrm{GMO}(X) = \mathrm{GO}(X)$ and $\mathrm{GMO}(Y) = \mathrm{GO}(Y)$, then by Theorem 5.2 we obtain the result of Theorem 3.4 of [37].

Lemma 5.2. (Noiri and Popa [38]) *If $f : (X, m_X) \to (Y, m_Y)$, where m_X has property B, is an M-continuous and M-closed injection and (Y, M_Y) is m-regular, then (X, m_X) is m-regular.*

Theorem 5.3. *Let (X, τ) and (Y, σ) be topological spaces and* GMO(X) *(resp.* GMO(Y)) *a gm-structure on X (resp. Y), where GMO(X) has property B. If $f : (X, \tau) \to (Y, \sigma)$ is a gM-continuous and gM-closed injection and (Y, σ) is gm-regular, then (X, τ) is gm-regular.*

Proof. The proof follows from Lemma 5.2.

Definition 5.7. A function $f : (X, \tau) \to (Y, m_Y)$ is said to be (τ, *m*)*-continuous* [43] if for each $x \in X$ and each $V \in m_Y$ containing $f(x)$, there exists an open set U containing x such that $f(U) \subset V$.

Definition 5.8. Let (X, τ) and (Y, σ) be topological spaces and GMO(Y) a *gm*-structure on Y. A function $f : (X, \tau) \to (Y, \sigma)$ is said to be (τ, *gm*)*-continuous* if $f : (X, \tau) \to (Y, \mathrm{GMO}(Y))$ is (τ, *m*)-continuous.

Lemma 5.3. (Noiri and Popa [38]) *If $f : (X, \tau) \to (Y, m_Y)$, where m_Y has property B, is a (τ, m)-continuous and m-closed surjection with compact point inverses and (X, τ) is regular, then (Y, m_Y) is m-regular.*

Theorem 5.4. *Let* (X, τ) *and* (Y, σ) *be topological spaces and* GMO(Y) *a gm-structure on* Y *having property B. If* $f : (X, \tau) \to (Y, \sigma)$ *is a* (τ, gm)*-continuous and gm-closed surjection with compact point inverses and* (X, τ) *is regular, then* (Y, m_Y) *is gm-regular.*

Proof. The proof follows from lemma 5.3.

Definition 5.9. An m-space (X, m_X) is said to be *m-normal* [38] if for each pair of disjoint m-closed sets F_1, F_2 of X, there exist $U_1, U_2 \in m_X$ such that $F_1 \subset U_1$, $F_2 \subset U_2$ and $U_1 \cap U_2 = \emptyset$.

Remark 5.5. Let (X, τ) be a topological space and $m_X = \tau$ (resp;. SO(X), PO(X), $\beta(X)$). Then m-normality coincides with normality (resp. semi-normality [13], pre-normality [39], β-normality [21]).

Definition 5.10. Let (X, τ) be a topologial space and GMO(X) a gm-structure on X. Then (X, τ) is sid to be *gm-normal* if (X, GMO(X)) is m-normal.

Lemma 5.4. (Noiri and Popa [38]) *If* $f : (X, m_X) \to (Y, m_Y)$, *where* m_X *and* m_Y *have property* $\mathcal{B}$, *is an M-closed and M-continuous surjection and* (X, m_X) *is m-normal, then* (Y, m_Y) *is m-normal.*

Theorem 5.5. *Let* (X, τ) *and* (Y, σ) *be topological spaces and* GMO(X) (*resp.* GMO(Y)) *a gm-structure on X (resp. Y) having property B. If* $f : (X, \tau) \to (Y, \sigma)$ *is a gM-continuous and gM-closed surjection and* (X, τ) *is gm-normal, then* (Y, σ) *is gm-normal.*

Proof. The proof follows from Lemma 5.4.

Lemma 5.5. (Noiri and Popa [38]) *If* $f : (X, m_X) \to (Y, m_Y)$, *where* m_X *has property a gm-structure on X (resp. Y), where* GMO(X) *has property B. If* $f : (X, \tau) \to (Y, \sigma)$ *is a gM-continuous and gM-closed injection and* (Y, σ) *is gm-normal, then* (X, τ) *is gm-normal.*

Proof. The proof follows from Lemma 5.5.

REFERENCES

1. M. E. Abd El-Monsef, S. N. El-Deeb and R. A. Mahmoud, *β-open sets and β-continuous mappings,* Bull. Fac. Sci. Assiut Univ. **12** (1983), 77-90.
2. M. E. abd El-Monsef, R. A. Mahmoud and E. R. Lashin, *β-closure and β-interior,* J. Fac. Ed. Ain Shams Univ. **10** (1986), 235-245.
3. D. Andrijević, *Semi-preopen sets,* Mat. Vesnik **38** (1986), 24-32.

4. I. Arokiarani, K. Balachandran and J. Dontchev, *Some characterizations of gp-irresolute and gp-continuous maps between topological spaces,* Mem. Fac. Sci. Kochi Univ. Ser. A Math. **20** (1999), 93-104.

5. M. Caldas, *Further results on generalized open mappings in topological spaces,* Bull. Calcutta Math. Soc. **88** (1996), 37-42.

6. M. Caldas, S. Jafari and T. Noiri, *Notions via g-open sets,* Kochi J. Math. **2** (2007), 43-50.

7. S. G. Crossley and S. K. Hildebrand, *Semi-closure,* Texas J. Sci. **22** (1971), 99-112.

8. R. Devi, H. Maki and K. Balachandran, *Semi-generalized closed maps and generalized semi-closed maps,* Mem. Fac. Sci. Kochi Univ. Ser. A Math. **14** (1993), 41-54.

9. R. Devi, K. Balachandran and H. Maki, *Semi-generalized homeomorphisms and generalized semi-homeomorphisms in topological spaces,* Indian J. Pure Appl. Math. **26** (1995), 271-284.

10. R. Devi, K. Balachandran and H. Maki, *On generalized α-continuous maps and α-generalized continuous maps,* Far East J. Math. Sci. Special Volume (1997), Part I, 1-15.

11. J. Dontchev, *On-generalizing semi-preopen sets,* Mem. Fac. Sci. Kochi Univ. Ser. A Math. **16** (1995), 35-48.

12. C. Dorsett, *Semi-regular spaces,* Soochow J. Math. **8** (1982), 45-53.

13. C. Dorsett, *Semi-normal spaces,* Kyungpook Math. J. **25** (1985), 173-180.

14. W. Dunham, *A new closure operator of non-T_1 topologies,* Kyungpook Mth. J. **22** (1982), 55-60.

15. W. Dunham and N. Levine, *Further results on generalized closed sets in topology,* Kyungpook Math. J. **20** (1980), 169-175.

16. S. N. El-Deeb, I. A. Hasanein, A. S. Mashhour and T. Noiri, *On p-regular spaces,* Bull. Math. Soc. Sci Math. R. S. Roumanie **27(75)** (1983), 311-315.

17. T. fukutake, a. A. Nasef and A. I. El-Maghrabi, *Some topological concepts via γ-generalized closed sets,* Bull. Fuknoka Univ. Ed. III **55** (2003), 1-9.

18. G. L. Garg and D. Sivaraj, *Presemiclosed mappings,* period. Math. Hungar. **19** (1988), 97-106.

19. N. Levine, *Semi-open sets and semi-continuity in topological spaces,* Amer. Math. Monthly **70** (1963), 36-41.

20. N. Levine, *Generalized closed sets in topology,* Rend. Circ. Mat. palermo (2) **19** (1970), 89-96.

21. R. A. Mahmoud and M. E. Abd El-Monsef, *β-irresolute and topological β-invariants,* Proc. Pakistan Acad. Sci. **27** (1990), 285-296.

22. H. Maki, K. Chandrasekhara Rao and A. Nagoor Gani, *On generalizing semi-open and preopen sets,* Pure Appl. Math. Sci. **49** (1999), 17-29.

23. S. R. Malghan, *Generalized closed maps,* J. Karnatak Univ. Sci. **27** (1982), 82-88.

24. a. S. Mashhour, M. E. Abd El-Monsef and S. N. El-Deep, *On precontinuous and weak precontinuous mappings,* Proc. Math. Phys, Soc. Egypt **53** (1982), 47-53.

25. A. S. Mashhour, I. A. Hasanein and S. N. El-Deeb, *α-continuous and α-open map-pings,* Acta Math. Hungar, **41** (1983), 213-218.

26. A. S. Mashhour, M. E. Abd El-Monsef and I. A. Hasanein, *On pretopological spaces,* Bull. Math. Soc. Math. R. S. roumanie **98(76)** (1984), 39-45.

27. M. Mocanu, *Generalizations of open functions,* Stud. Cerc. St. Ser. Mat., Univ. Bacau, **13** (2003), 67-78.

28. B. M. Munshi, *Separation axioms,* Acta Ciencia Indica **12** (1986), 140-144.

29. B. M. Munshi and D. S. Bassan, *g-continuous mappings,* Vidya J. guadajarat Univ. B. Sci. **24** (1981), 63-68.

30. G. B. Navalagi, *Quasi α-closed, strongly α-closed, weakly α-irresolute mappings,* Topology Atlas #421.

31. O. Njåstad, *On some classes of nearly open sets,* Pacific J. Math. **15** (1965), 961-970.

32. T. Noiri, *A generalization of closed mappinjgs,* Atti accad. Naz. Lincei Rend. Cl. Sci. Fiz. Mat. Natur. (8) **54** (1973), 412-415.

33. T. Noiri, *Almost g-closed functions and separation axioms*, Acta Math. Hungar. **82** (1999), 193-205.

34. T. Noiri, *Weak and strong forms of β-irresolute functions.* Acta Math. Hungar. **99** (2003), 315-328.

35. T. Noiri, *A unified theory of certain modifications of generalized closed sets,* Internat. J. General Topology **1** (2008), 87-99.

36. T. Noiri, H. Maki and J. Umehara, *Generalized preclosed functions,* Mem. Fac. Sci. Kochi Univ. Ser. A Math **19** (1998), 13-20.

37. T. Noiri and V. Popa, *On g-regular spaces and some functions,* Mem. Fac. Sci. Kochi Univ. Ser. A Math. **20** (1999), 67-74.

38. T. Noiri and V. Popa, *A unified theory of closed functions,* Bull. Math. Soc. Sci. Math. Roumanie **49(97)** (2006), 371-382.

39. T. M. Nour, *Contributions to Theory of Bitopological Spaces,* Ph. D. Thesis, Univ. of delhi, 1989.

40. M. C. pal and P. Bhattacharyya, *Feeble and strong forms of preirresolute functions,* Bull. Malaysian Math. Sci. Soc. (2) **19** (1996), 63-75.

41. V. Popa and T. Noiri, *On M-continuous functions*, Anal. Univ. "dunarea de Jos" Galati, Ser. Mat. Fiz. Mec. Teor., fasc. II **18(23)** (2000), 31-41.

42. V. Popa and T. Noiri, *On the definitions of some generalized forms of continuity under minimal conditions,* Mem. Fac. Sci. Kochi Univ. Ser. Math. **22** (2001), 31-41.

43. V. Popa and T. Noiri, *On weakly* (τ, *m*)*-continuous functions,* Rend. Circ. Mat. Palermo (2) **51** (2002), 295-316.

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REFERENCES

1. N. L. Alling, Foundations of Analysis on Surreal Number Fields, North-Holland Publishing Co., 1987.
2. E. Hewitt, Rings of continuous functions I, Trans. Amer. Math. Soc. 64(1948), 54-99.

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